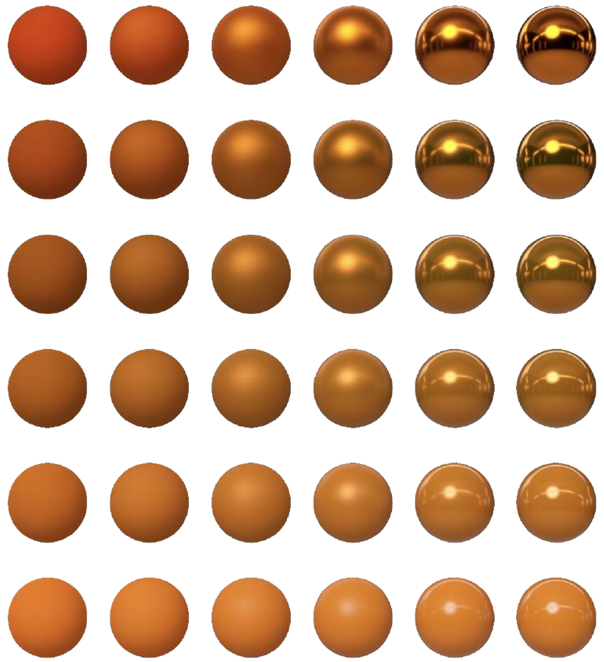
Advanced Computer Graphics
Physically-Based Lighting/Rendering



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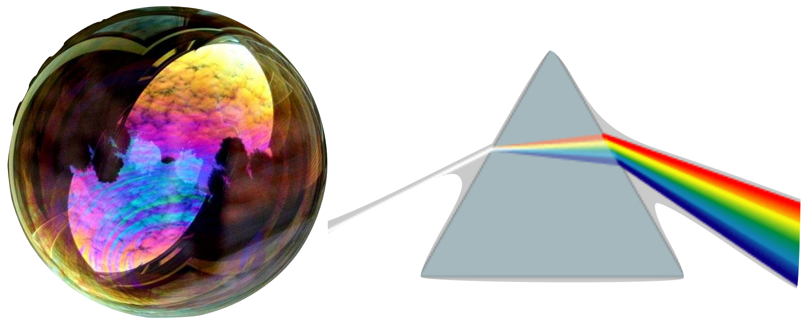
Radiometry: Physically Measuring Light

- Interaction of light with objects much larger than its wavelength
- Geometric optics suffices for most cases in computer graphics
  - No interference, no dispersion, etc.



- Advantages:
  - Linearity: effect(light 1 + light 2) = effect(light 1) + effect(light 2)
  - Energy conservation
  - No polarization
  - Lights of different wavelengths are independent of each other (no fluorescence)

Four Radiometric Quantities

- Flux: how much energy flows from/to/through a surface per unit of time
  - Think of it as "photons per second"
  - Symbol: Φ , units = Watt
  - Example light bulb: surface = sphere around light bulb
    - Radius doesn't matter: #photons crossing the boundary of the sphere is the same
  - Example table surface: doesn't matter where photons come from

- Irradiance: energy density = flux per area

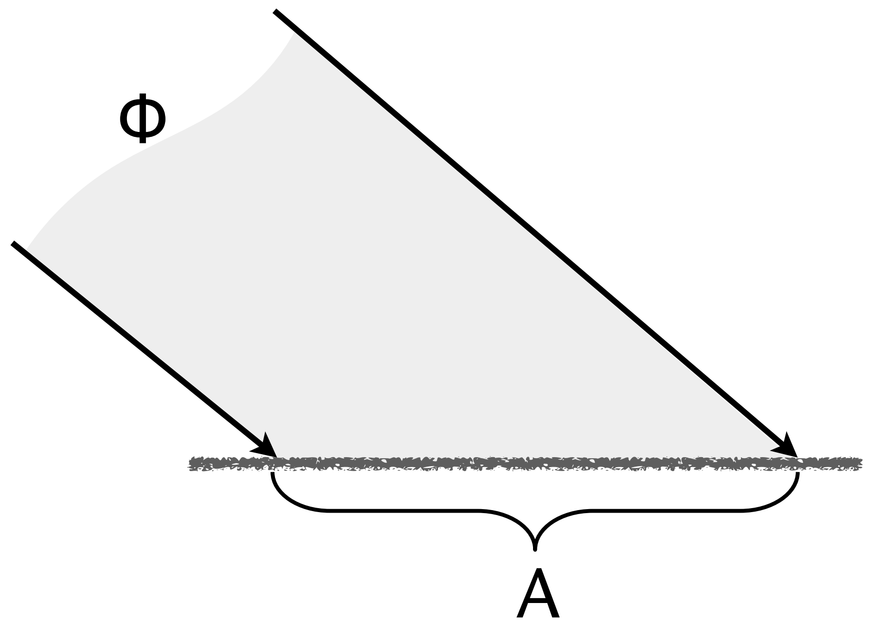
$$E = \frac{d\Phi}{dA}$$

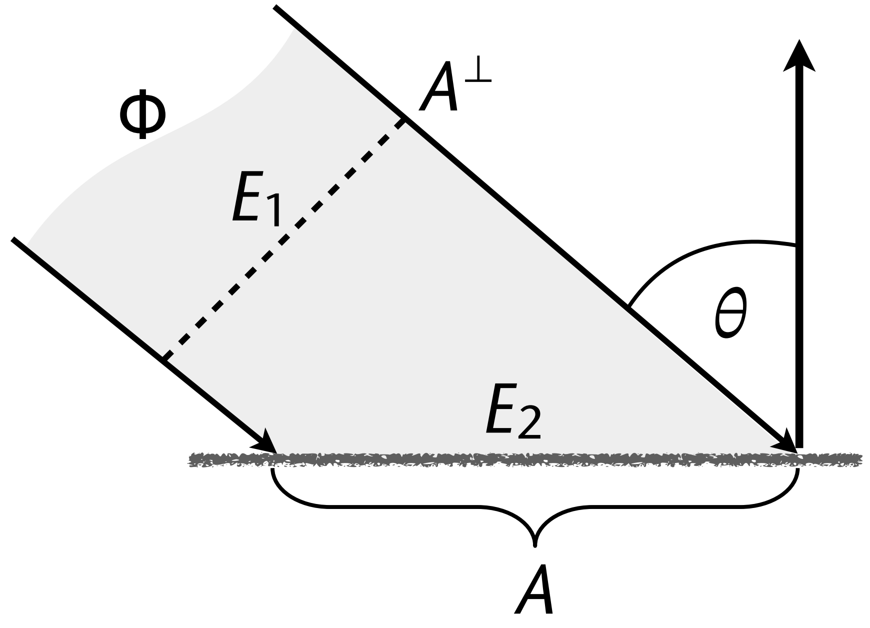
- Example: 50 Watts incident on table surface of 2m²
-> each point on the table surface receives irradiance of 25 Watt/m²

- Lambert's law: irradiances on surfaces are related by the cos(θ) factor

$$E_1 = \frac{d\Phi}{dA^\perp} \qquad E_2 = \frac{d\Phi}{dA} = \frac{d\Phi \cos\theta}{dA^\perp} = E_1 \cos\theta$$

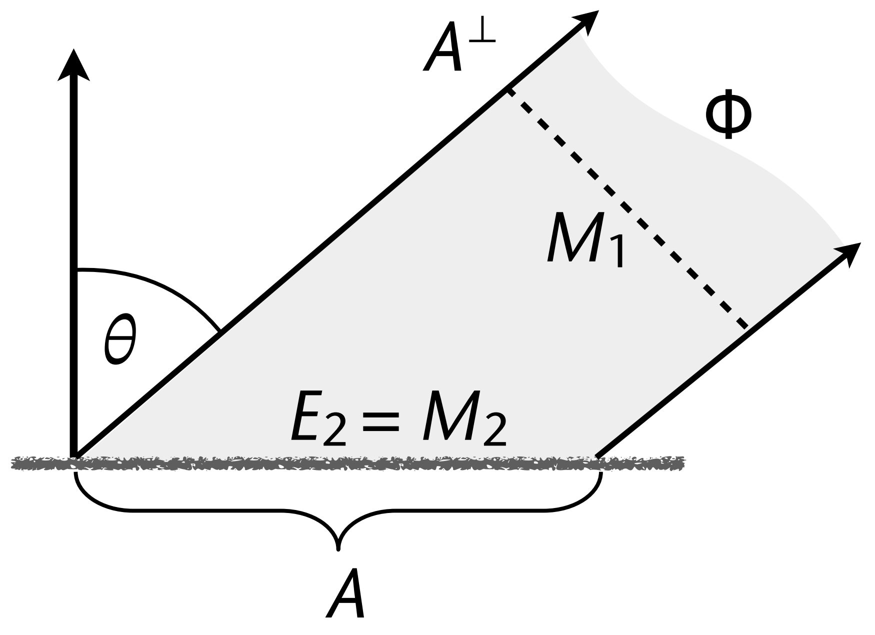
since $A^\perp = A\cos\theta$





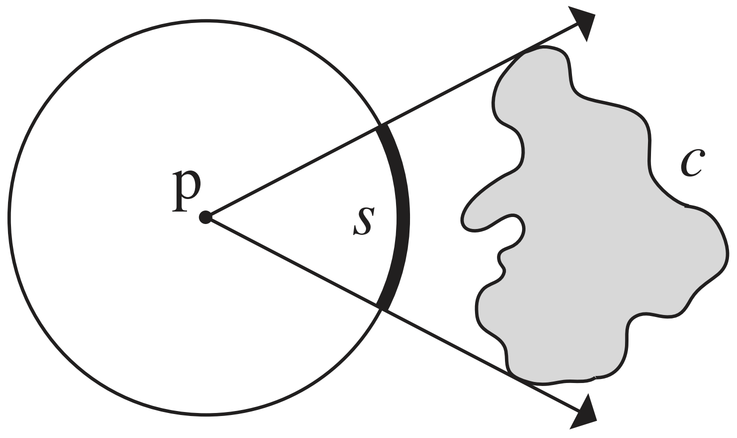
- Exitance (a.k.a. exiting irradiance, radiosity): same thing as irradiance, except for light *leaving* a surface

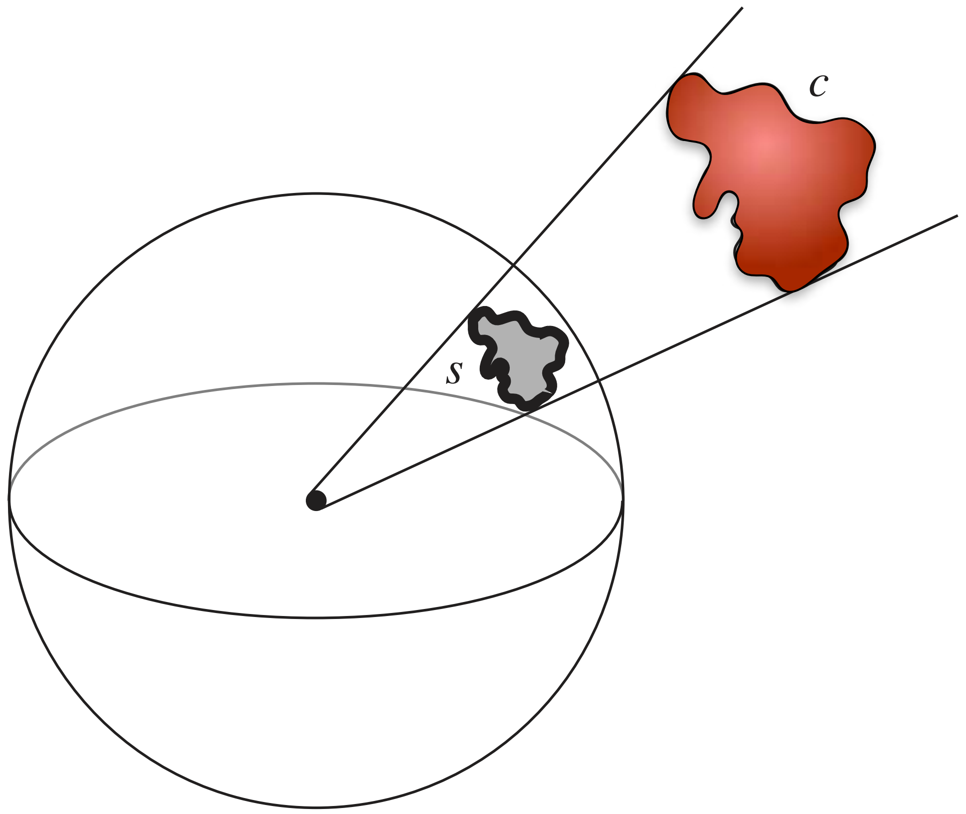
$$M_1 = \frac{d\Phi}{dA^{\perp}} = \frac{d\Phi}{\cos\theta\, dA} = \frac{E_2}{\cos\theta}$$



Solid Angles

- Generalization of *angles* to 3D:
  - Analog: subtended angle of an object in 2D = length of arc of its projection on the unit circle
    - Units = radians (rad), range = $[0,2\pi]$
  - Subtended solid angle of an object in 3D = area of its projection on the unit sphere
    - Units = steradians (sr), range = $[0,4\pi]$
    - Solid angle = area on the unit sphere





- Notation:
  - ω = direction = unit vector = $\begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi \sin\theta \\ \cos\theta \end{pmatrix}$
  - $d\omega$ = infinitesimal solid angle in direction ω
  - Sometimes, ω is used to denote a solid angle, too

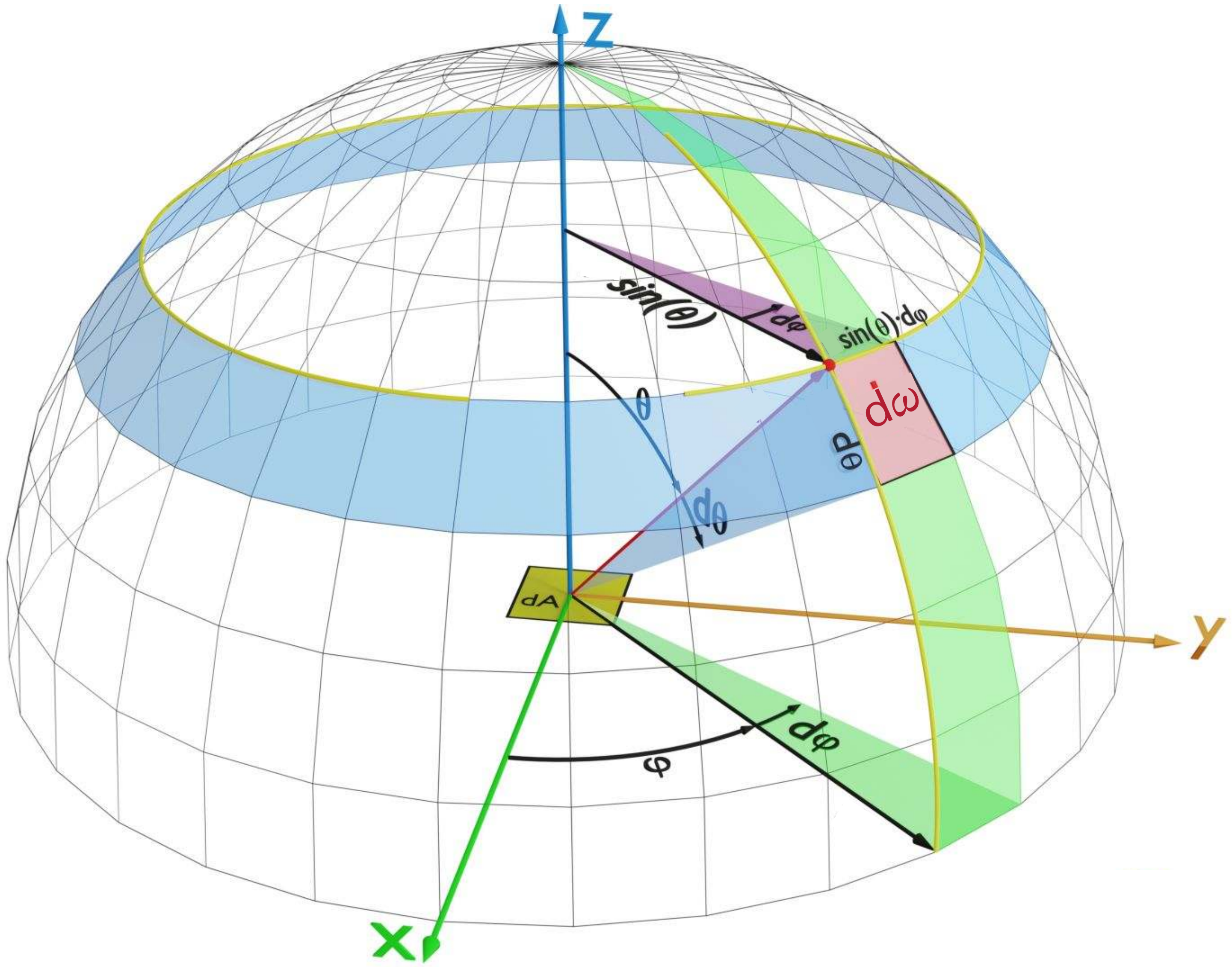
Note on the Side: How to Make a Change of Variables in Integrations

- When evaluating integrals over the hemisphere, Ω, we could use the solid angle directly, or polar angles instead
  - Polar angles: $\varphi \in [0, 2\pi]$, $\theta \in [0, \pi/2]$ where $\theta = 0$ points *upwards*
  - Consider a square solid angle, dω :

$$\begin{aligned} d\omega &= \text{width} \cdot \text{height} \\ &= (\sin\theta \cdot d\varphi) \cdot d\theta \end{aligned}$$

  - Therefore,

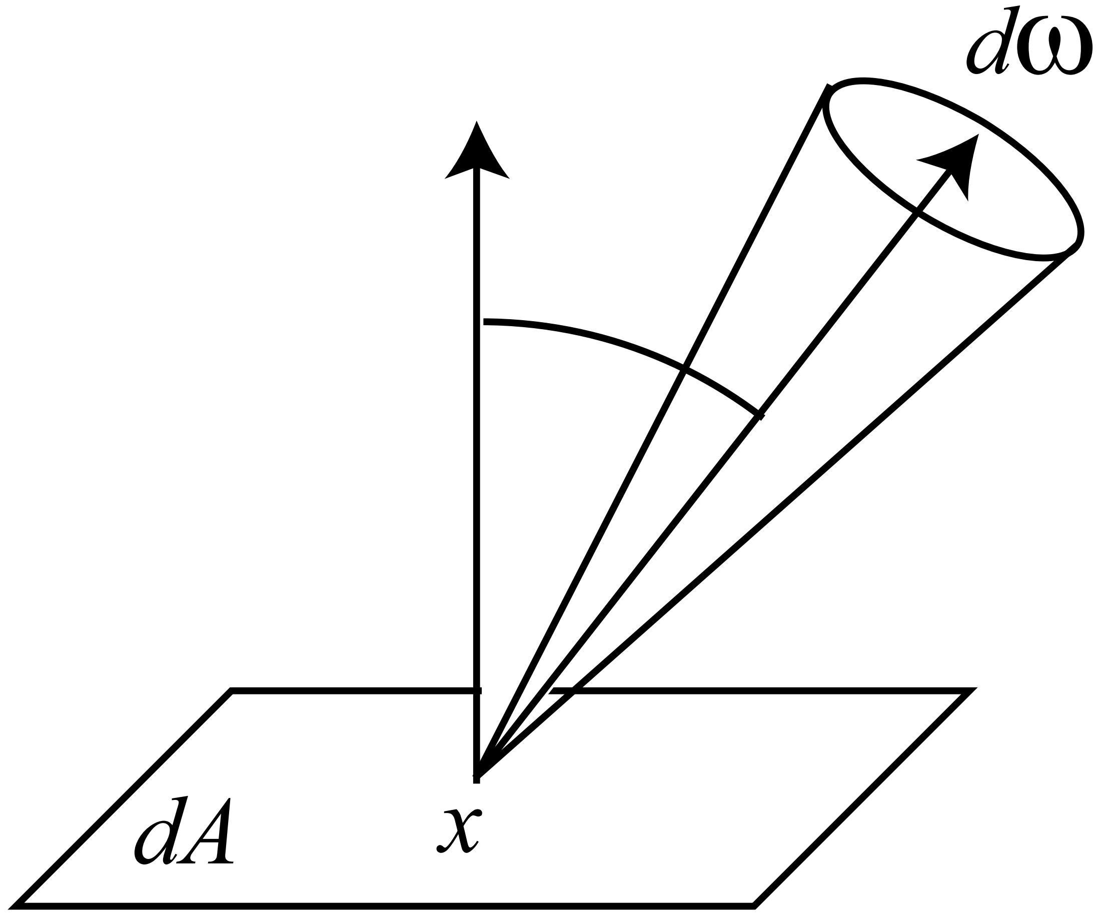
$$\int_\Omega \ldots d\omega = \int_0^{\pi/2} \int_0^{2\pi} \ldots \sin(\theta)\, d\varphi\, d\theta$$



- Intensity: flux over solid angles

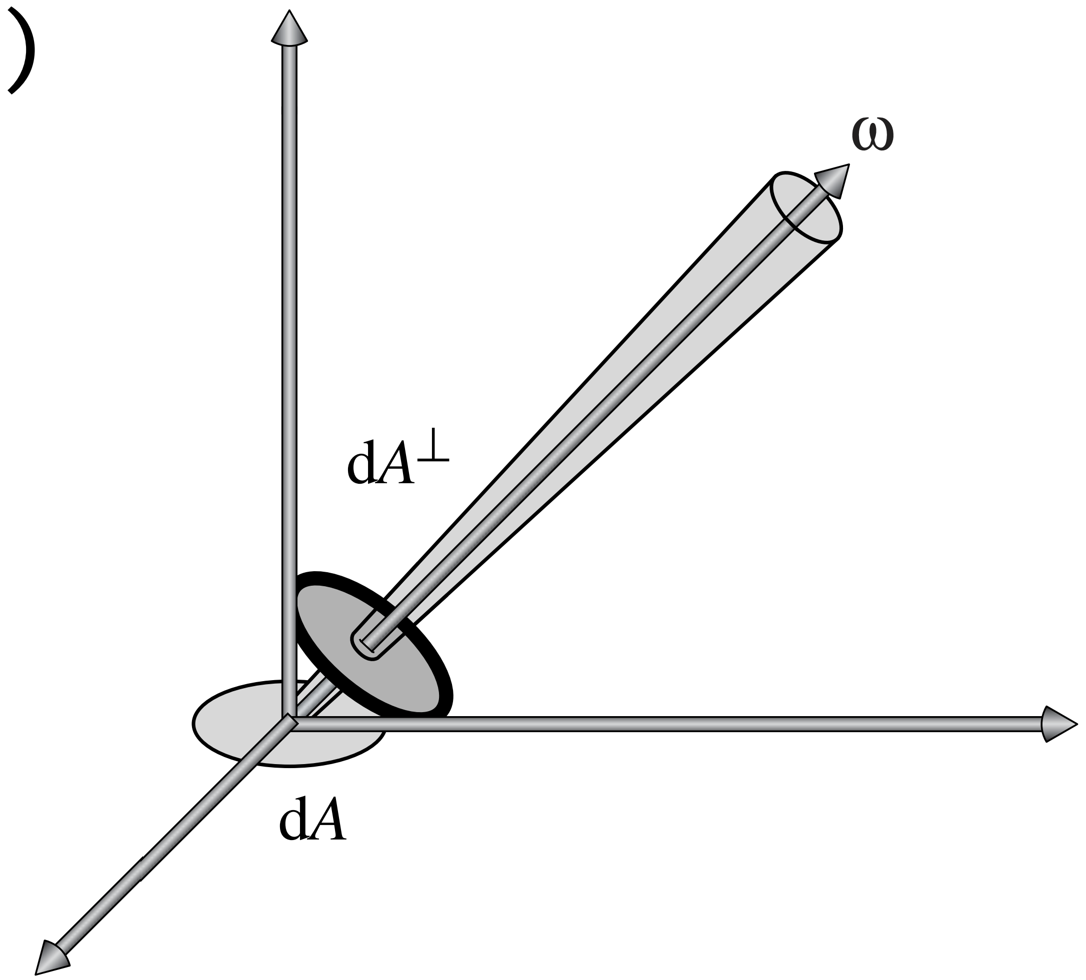
$$I = \frac{\mathrm{d}\Phi}{\mathrm{d}\omega}$$

- Think "number of photons sent into a small solid angle around direction ω"
- Only meaningful for a outgoing light emanating from a single pont **x** , or light coming in from dω and converging in point **x**
- Like irradiance, it is a density, but a different kind of energy density!



- Radiance: "how much power arrives at (or leaves from) a specific point on a surface, per unit solid angle, *and* per unit projected area"

$$L = \frac{d^2\Phi}{d\omega\, dA^{\perp}}$$



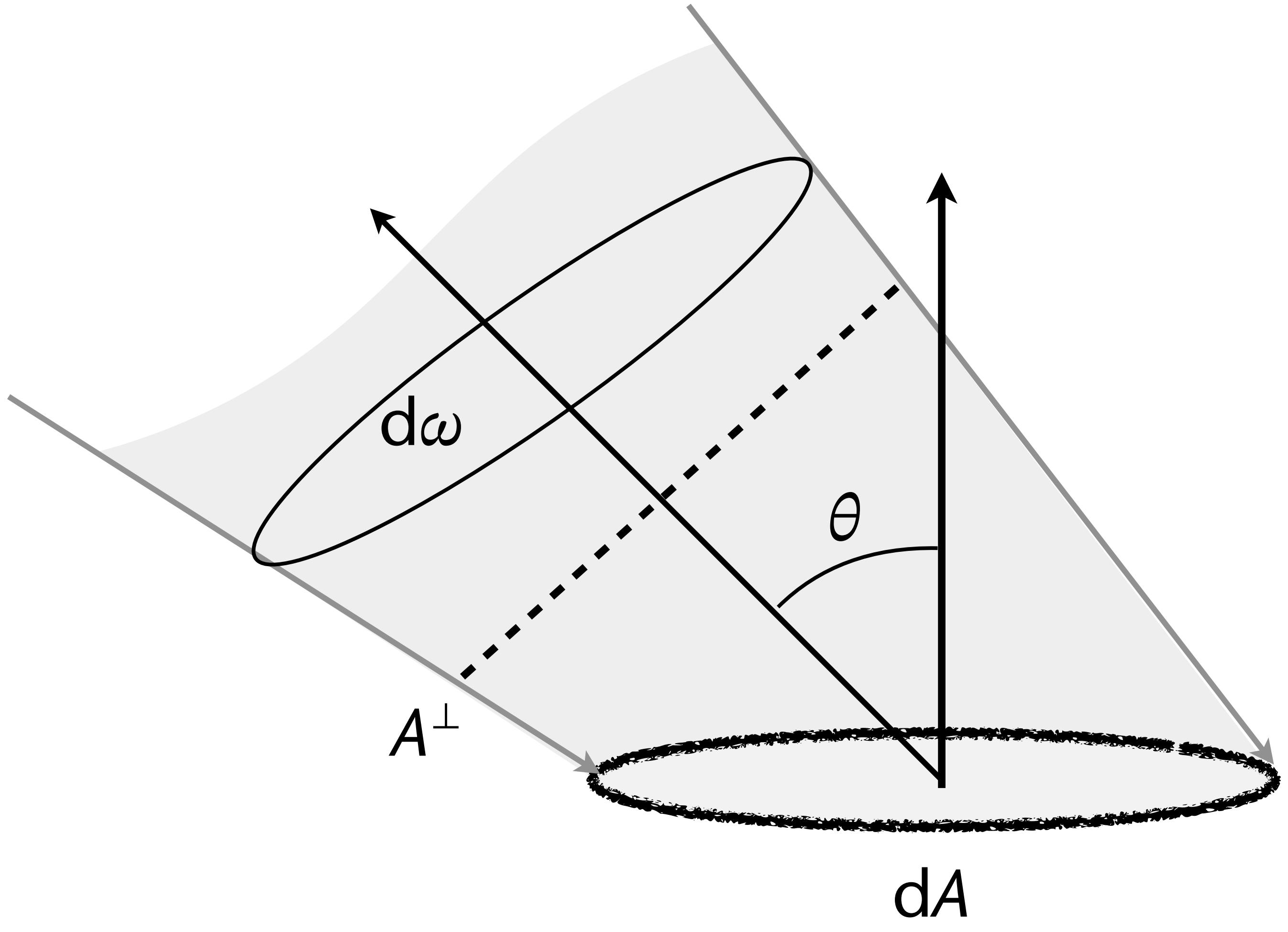
- Think: "limit of incident light at the surface as a cone of incident directions, dω, becomes very small, and as the local area on the surface, dA, also becomes very small"
- Think: combination of 2 densities, *irradiance* and *intensity*
- Think: a neuron on the retina, or a pixel on a CCD chip inside a camera, measures *radiance* (at a specific small area, arriving from a specific small cone)

Simple Properties and Relationships of Radiance

- Relationships between L, E, I:

$$L = \frac{d^2\Phi}{d\omega\, dA^\perp} = \frac{dE}{d\omega} = \frac{dI}{dA^\perp}$$

$$L = \frac{d^2\Phi}{\cos\theta\, dA\, d\omega}$$

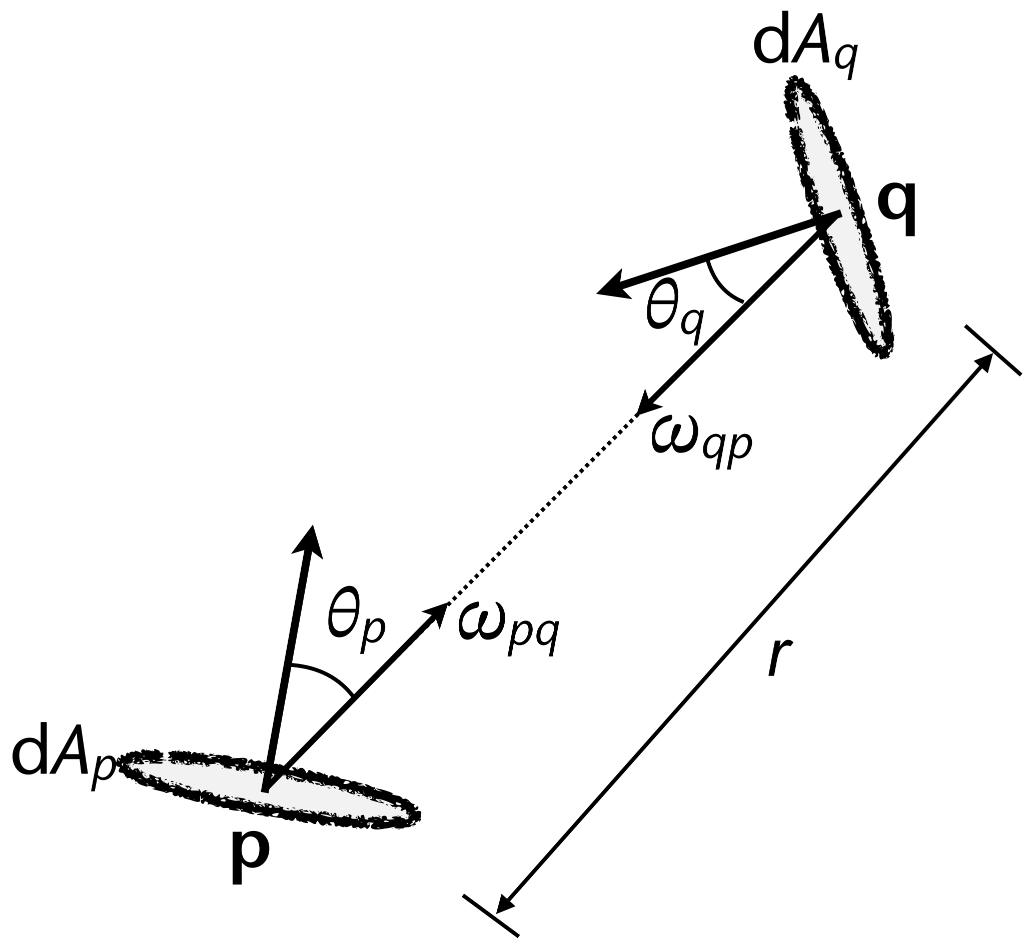


- Notes:
  - L is a 5-dimensional function $L(p, \omega)$
  - Radiance is defined both for *incoming*, $L_i(p, \omega)$, as well as *outgoing* light, $L_o(p, \omega)$!
  - Direction ω always points *away* from point p
  - For incoming light (L_i): $\cos\theta = \mathbf{n}\cdot\boldsymbol{l}$, for outgoing light (L_o): $\cos\theta = \mathbf{n}\cdot\mathbf{v}$

Important Property: Radiance is Invariant Along Straight Paths

- Arguably the most important property
- Consider light that travels from point **p** to point **q** (without any participating media)
- Exiting radiance

$$L_o(\omega_{pq}) = \frac{\mathrm{d}^2 \Phi_{pq}}{\cos\theta_p \, \mathrm{d}A_p \, \mathrm{d}\omega_{pq}} \qquad (1)$$



- Incident radiance

$$L_i(\omega_{qp}) = \frac{\mathrm{d}^2 \Phi_{qp}}{\cos\theta_q \, \mathrm{d}A_q \, \mathrm{d}\omega_{qp}} \qquad (2)$$

- No light is lost along the path (i.e., we assume vacuum), therefore

$$\mathrm{d}^2 \Phi_{pq} = \mathrm{d}^2 \Phi_{qp} \qquad (3)$$

- Solve (1) and (2) for $d^2\Phi$ and plug into (3):

$$L_o(\omega_{pq}) \cos\theta_p \, \mathrm{d}A_p \, \mathrm{d}\omega_{pq} = L_i(\omega_{qp}) \cos\theta_q \, \mathrm{d}A_q \, \mathrm{d}\omega_{qp} \qquad (4)$$

- Compute $\mathrm{d}\omega_{pq} = \frac{\mathrm{d}A^{\perp}}{r^2} = \frac{\cos\theta_q \, \mathrm{d}A_q}{r^2}$, and similarly $\mathrm{d}\omega_{qp}$, and plug into (4)
- Thus,

$$L_o(\omega_{pq}) \cos\theta_p \, \mathrm{d}A_p \frac{\cos\theta_q \, \mathrm{d}A_q}{r^2} = L_i(\omega_{qp}) \cos\theta_q \, \mathrm{d}A_q \frac{\cos\theta_p \, \mathrm{d}A_p}{r^2}$$

- And, hence,

$$L_o(\omega_{pq}) = L_i(\omega_{qp})$$

- Thus, *radiances are the physically correct quantities to exchange* between points!

Reasons for Radiance

- Eyes / cameras measure radiance
- Radiance is invariant w.r.t. distance
- Therefore, physically-correct lighting must compute radiances
-

Determining Useful Quantities from Radiance

- Computing the total irradiance at a point on a surface:
  - Irradiance from a specific direction:

$$L_i(\omega) = \frac{\mathrm{d}^2\Phi(\omega)}{\cos\theta\,\mathrm{d}A\,\mathrm{d}\omega} \qquad L_i(\omega)\cos\theta\,\mathrm{d}\omega = \frac{\mathrm{d}\Phi(\omega)}{\mathrm{d}A} = E(\omega)$$

  - Needs to be integrated over all directions to obtain the total "brightness" (Watt/m²):

$$E = \frac{\mathrm{d}\Phi}{\mathrm{d}A} = \int_\Omega L_i(\omega)\cos\theta\,\mathrm{d}\omega$$

where Ω is the hemisphere centered on the normal in point p

- Irradiance at a point **p** arriving from a distant surface *A*:
  - From the definition of radiance:

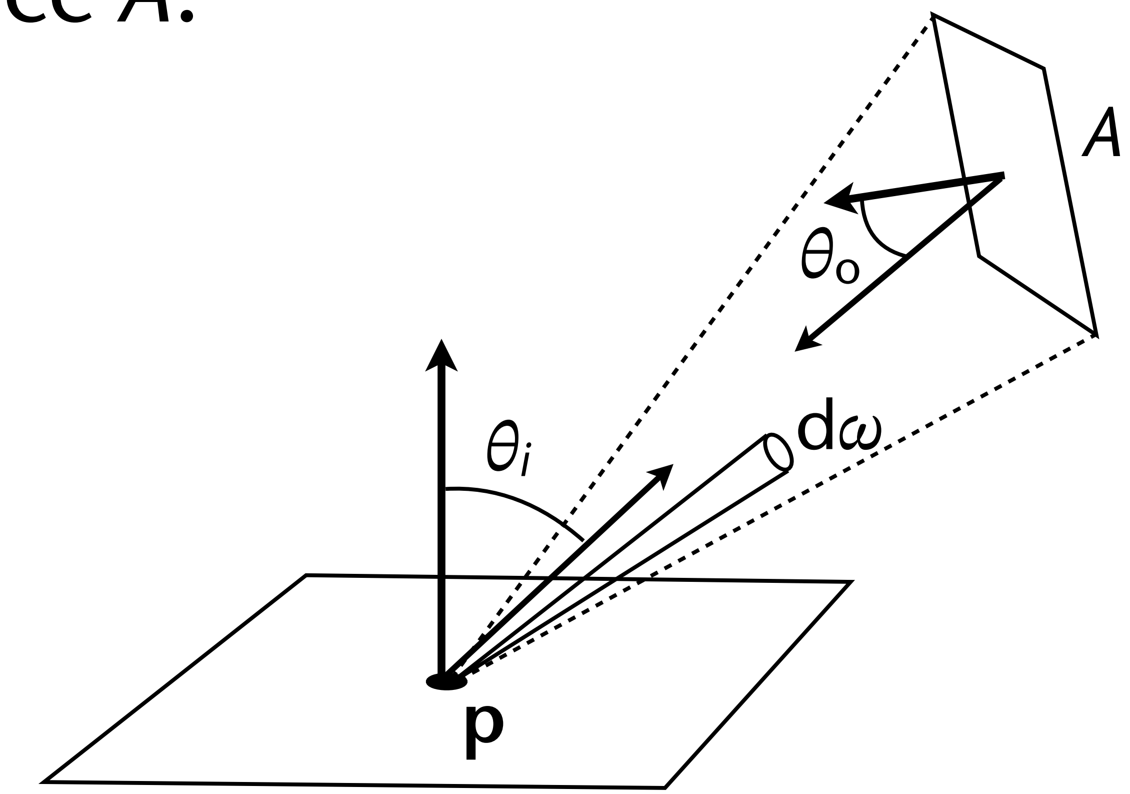
$$dE = L \cos\theta_i \, d\omega$$

  - Integrating leads to:

$$E = \int_{\Omega'} L(\theta_i, \varphi_i) \cos\theta_i \, d\omega$$

  - However, this would need to be integrated over that part of Ω where A is visible $\rightarrow \Omega'$, which is cumbersome
  - Trick: replace $d\omega$ by $d\omega = \cos\theta_o \cdot \frac{dA}{r^2}$
  - Thus,

$$E = \int_A L(\omega_i) \cos\theta_i \cos\theta_o \frac{1}{r^2} \, dA$$



- Total flux (power) emitted from a surface:
  - Consider the outgoing radiance from a surface A

$$L_o(\omega) = \frac{\mathrm{d}^2\Phi(\omega)}{\cos\theta\,\mathrm{d}A\,\mathrm{d}\omega} \qquad\qquad L_o(\omega)\cos\theta\,\mathrm{d}A\,\mathrm{d}\omega = \mathrm{d}^2\Phi(\omega)$$

  - Note that θ is one of the polar angles of ω, and $\cos\theta = \mathbf{n}\cdot\mathbf{v}$, where **v** points in the direction of ω
  - Integrating over all directions and the whole surface yields

$$\Phi_o = \int_A \int_\Omega L_o(\omega)\cos\theta\,\mathrm{d}\omega\,\mathrm{d}A$$

- Except: we don't know L_o ... (yet)

Example

- Consider area light source A = 10x10 cm²
- Assume, all points on A have radiance $L_o(\phi, \theta) = 6000 \cos\theta \frac{W}{sr \cdot m^2}$
- Exiting power of the light source:

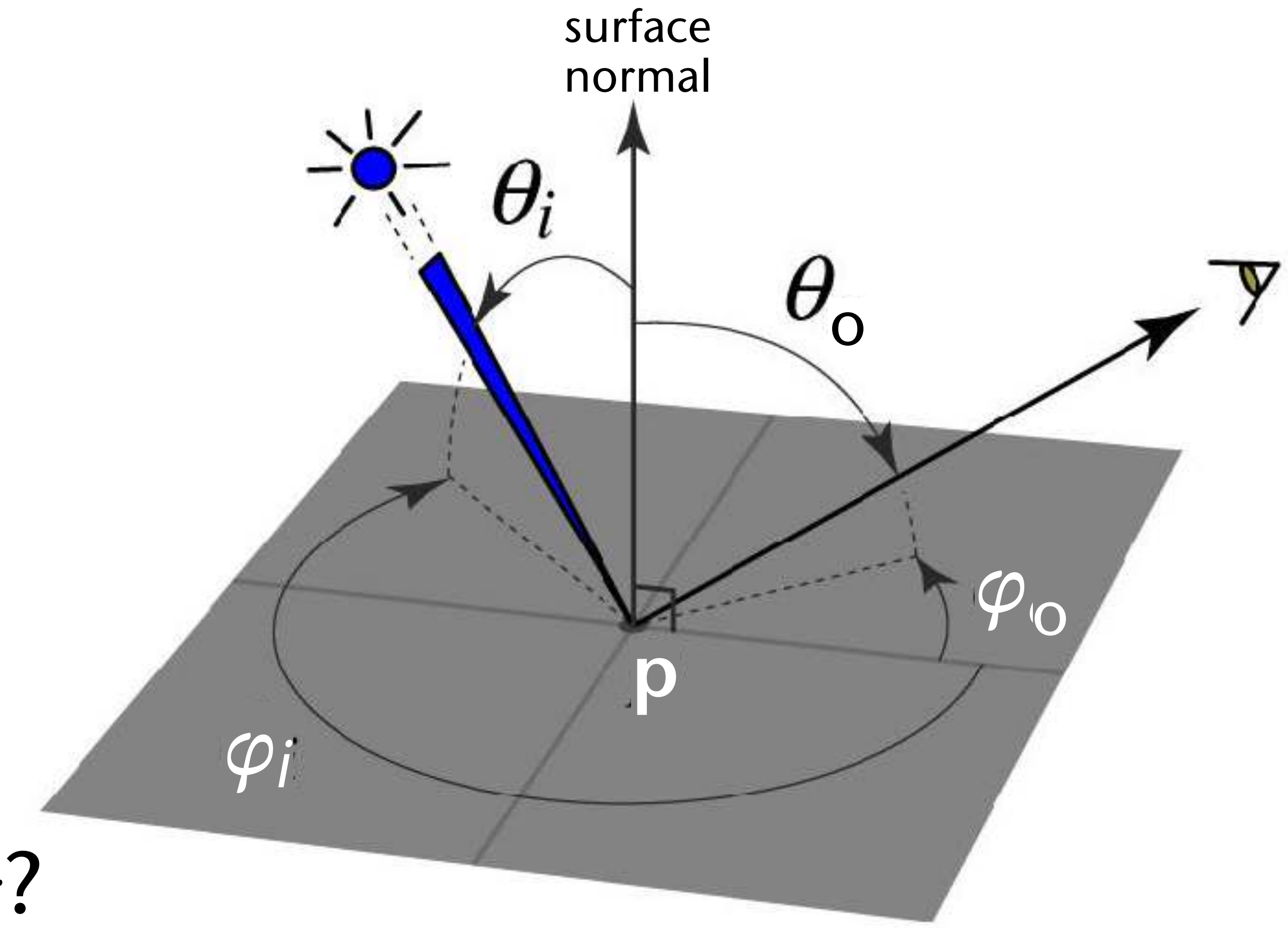
$$\Phi = \int_A \int_\Omega L_o \cos\theta \, \mathrm{d}\omega \, \mathrm{d}A \quad = \int_A \int_\Omega 6000 \cos^2\theta \, \mathrm{d}\omega \, \mathrm{d}A$$

$$= \int_A 6000 \int_0^{2\pi} \int_0^{\pi/2} \cos^2\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \, \mathrm{d}A \quad = \int_A 6000 \cdot 2\pi \cdot \left[\frac{-\cos^3\theta}{3} \right]_0^{\pi/2} \mathrm{d}A$$

$$= 6000 \cdot 2\pi \cdot \frac{1}{3} \frac{W}{m^2} \cdot 0.1m \cdot 0.1m \quad \approx 125\ W$$

The BRDF: Interaction of Light with Surfaces

- Given some light shining on point **p** on a surface, coming in from direction ω_i , how much light exits that point into another direction ω_o ?
- Assumptions:
  - Light does not change wavelength
  - Reflection / scattering is instantaneous
  - Light does not travel inside the material (no subsurface scattering)
- Question: how much radiance is exiting in direction ω_o, as a result of incident radiance coming from direction ω_i?



Definition of BRDF

- Irradiance at point **p** depends on incident radiance, L_i:

$$\mathrm{d}E(\omega_i) = L_i(\omega_i)\cos\theta_i\,\mathrm{d}\omega_i$$

- Some of that irradiance gets scattered into direction ω_o, leading to exiting radiance L_o
- By the linearity assumption: $\mathrm{d}L_o(\omega_o) \propto \mathrm{d}E(\omega_i)$
- The proportionality factor is defined as the Bidirectional Reflectance Distribution Function (BRDF)

$$\rho(\omega_o, \omega_i) = \frac{\mathrm{d}L_o(\omega_o)}{\mathrm{d}E(\omega_i)} = \frac{\mathrm{d}L_o(\omega_o)}{L_i(\omega_i)\cos\theta_i\,\mathrm{d}\omega_i}$$

Note on Notations of the BRDF Function

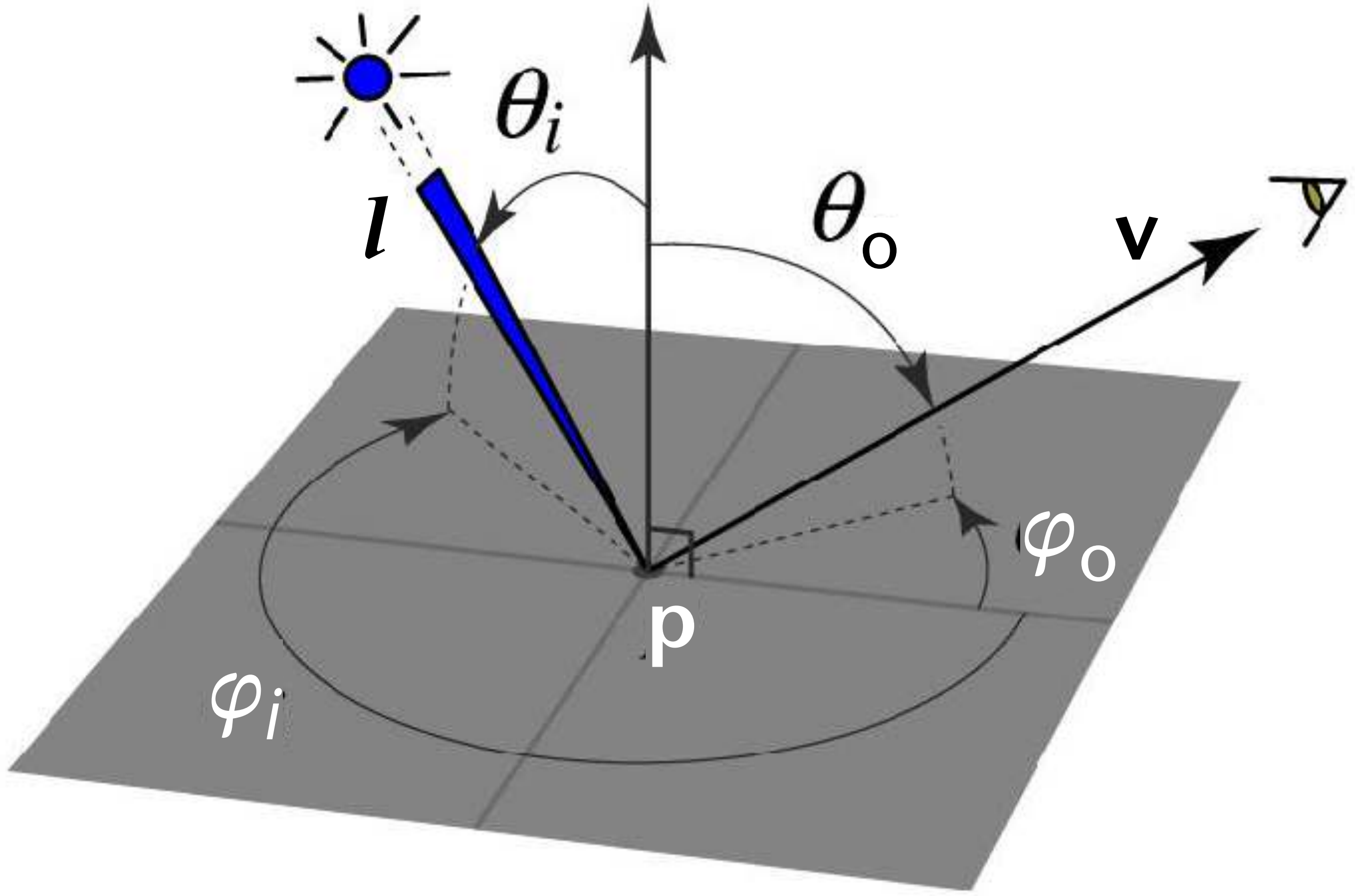
- The BRDF function is a 4D function:

$$\rho(\omega_o, \omega_i) = \rho(\theta_o, \varphi_o, \theta_i, \varphi_i)$$

- Assumes that ρ is constant over the surface
- Sometimes, ρ is written in terms of vectors:

$$\rho = \rho(\boldsymbol{l}, \mathbf{v})$$

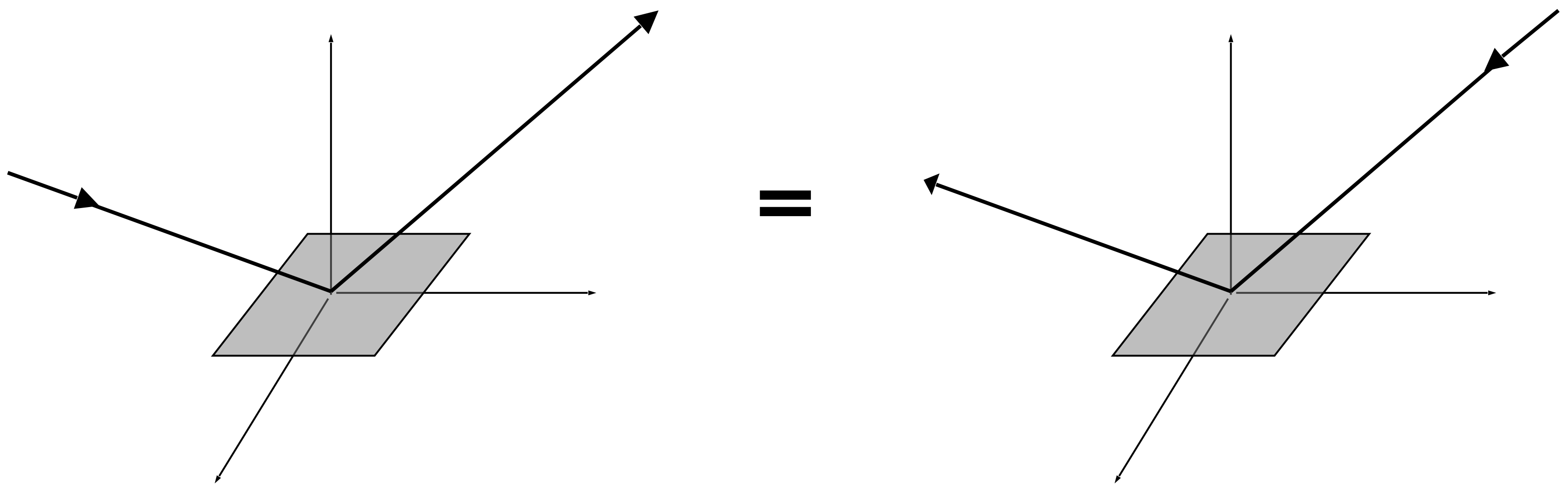
  - In practice, this version is preferred,
    since cos(angle) = scalar product of vectors



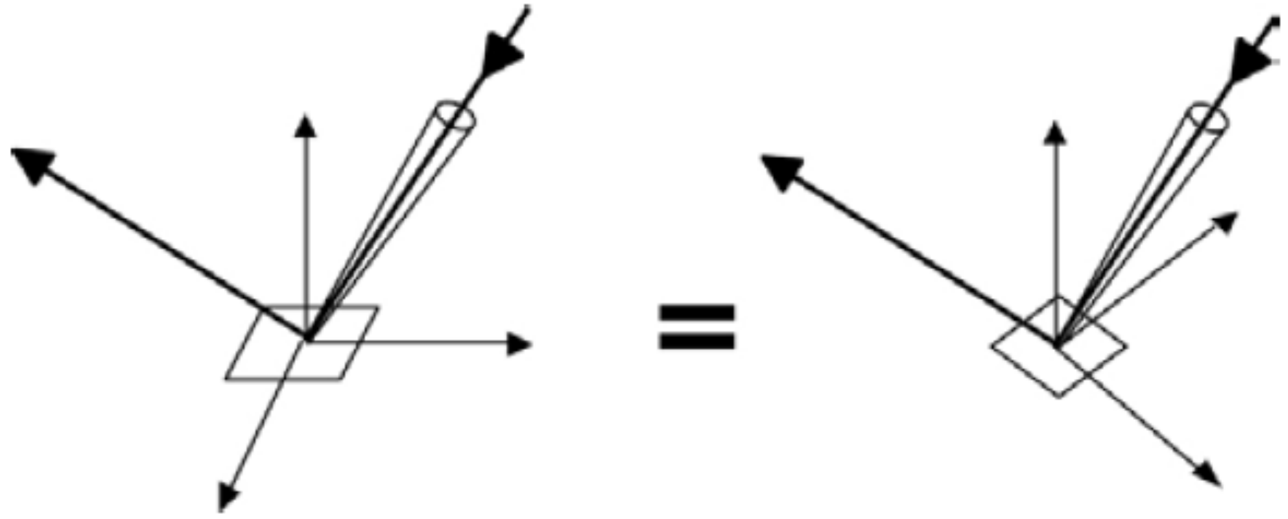
Properties of the BRDF

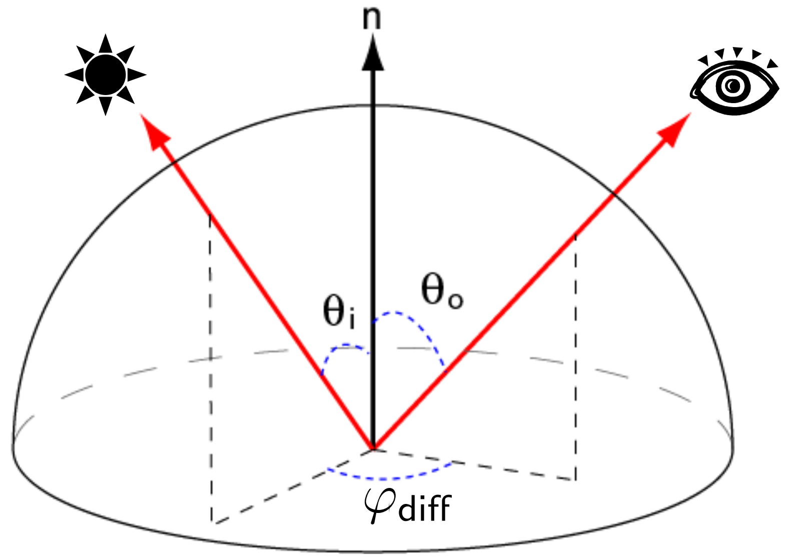
- Reciprocity:

$$\rho(\omega_i, \omega_o) = \rho(\omega_o, \omega_i)$$



- In general, the BRDF is anisotropic
  - Examples: brushed metal, hair, satin, ...
- For many materials, though, it is isotropic
  - Consequence, $\rho = \rho(\theta_i, \theta_o, \varphi_{\text{diff}})$
- Positivity: $\rho \geq 0$ everywhere





The Reflectance Equation (aka Rendering Equation)

- Computing the outgoing radiance
- Solve definition of BRDF for L_o:

$$\mathrm{d}L_o(\omega_o) = \rho(\omega_o, \omega_i) L_i(\omega_i) \cos\theta_i \, \mathrm{d}\omega_i$$

- Integrate over all incoming directions:

$$L_o(\omega_o) = \int_\Omega \rho(\omega_o, \omega_i) L_i(\omega_i) \cos\theta_i \, \mathrm{d}\omega_i$$

- In practice, we sample into "all" possible directions ω_i:
  - Many different, clever methods to choose the "best" sampling directions (Monte Carlo, variance reduction, importance-based sampling)

- Alternative "spelling" of the reflectance equation:

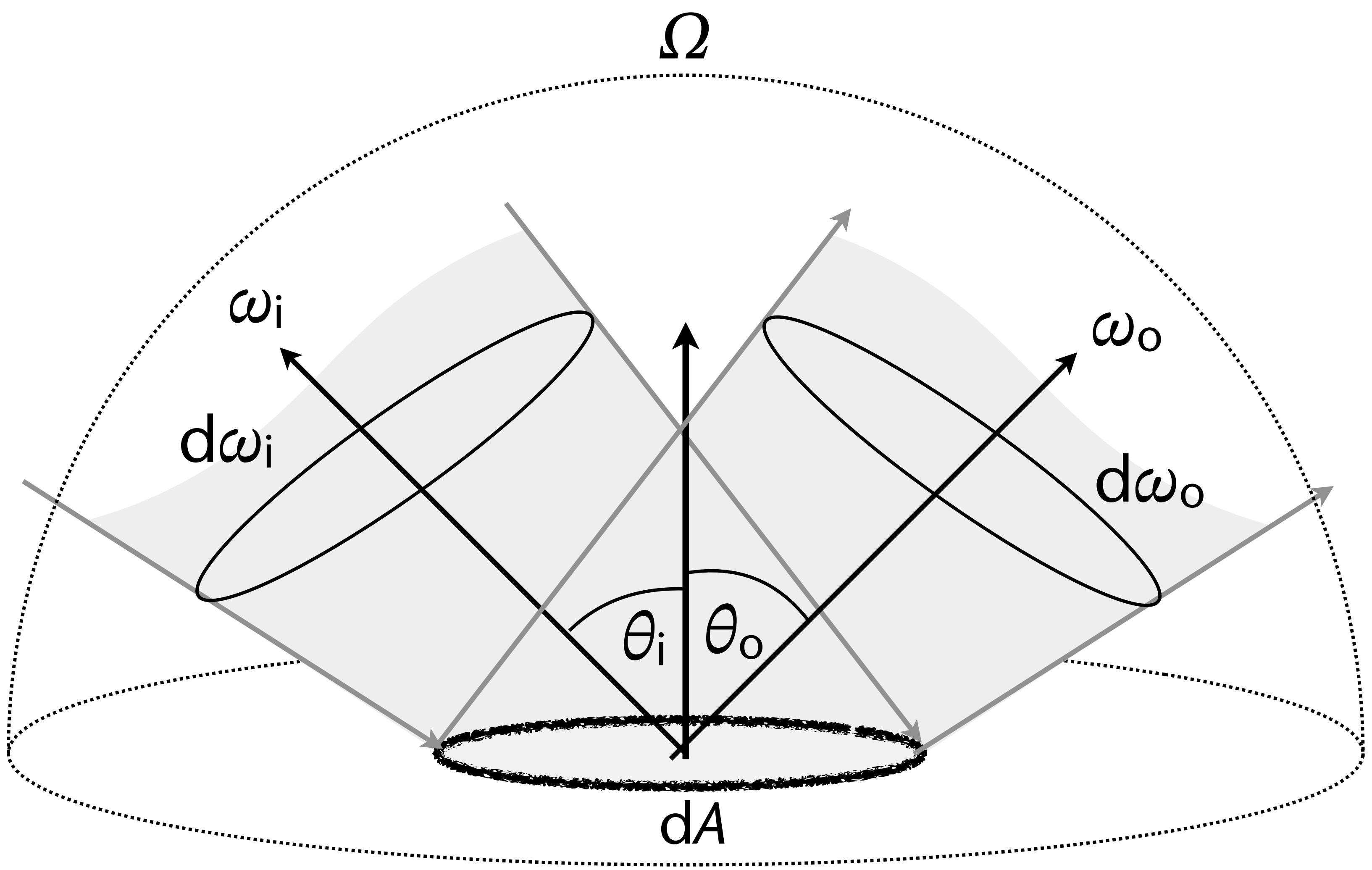
$$L_o(\mathbf{v}) = \int_\Omega \rho(\boldsymbol{l}, \mathbf{v}) \cdot L_i(\boldsymbol{l})(\mathbf{n} \cdot \boldsymbol{l})\, \mathrm{d}\omega_i$$

- Note that the term $\cos\theta_i = \mathbf{n} \cdot \boldsymbol{l}$ is *not* part of the BRDF!

Another Property of BRDFs: Conservation of Energy

- No energy can be *created*
  - (Also, it cannot be destroyed, but that is usually modeled differently in CG)
- Incoming (incident) irradiance from all directions:

$$E = \int_{\Omega} L_i(\omega_i) \cos\theta_i \, d\omega_i$$

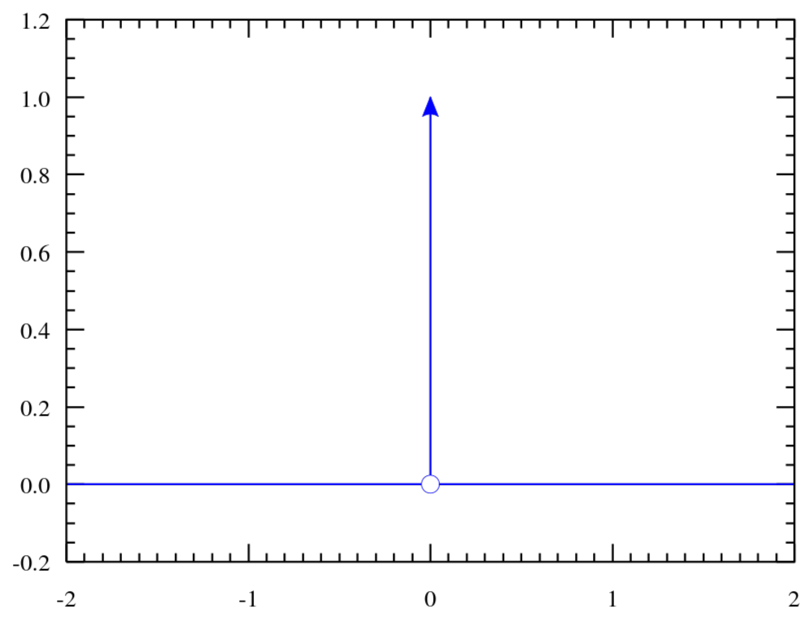


- Outgoing (exitant) irradiance in all dir's:

$$M = \int_{\Omega} L_o(\omega_o) \cos\theta_o \, d\omega_o = \int_{\Omega}\int_{\Omega} \rho(\omega_o, \omega_i) L_i(\omega_i) \cos\theta_i \, d\omega_i \, \cos\theta_o \, d\omega_o$$

- Because of conservation of energy, $M \overset{!}{\leq} E$, i.e., $\frac{M}{E} \overset{!}{\leq} 1$ (1)
  for all and *any* L_i (!)
- Note: $\frac{M}{E}$ is sometimes called "reflectance"
  - Meaning: how much of all incoming light (from all directions) is reflected (scattered) into all directions
- Choose the special incoming radiance $L_i(\omega_i) = \bar{L}_i \delta(\omega_i - \bar{\omega})$
  where δ is the Dirac function

$$\delta(x) \text{"="} \begin{cases} +\infty & , x = 0 \\ 0 & , x \neq 0 \end{cases}$$



  - Important property:

$$\int f(x)\delta(x - \bar{x})\,\mathrm{d}x = f(\bar{x})$$

- With this special L_i, the incoming irradiance becomes

$$E = \int_\Omega \bar{L}_i \delta(\omega_i - \bar{\omega}) \cos\theta_i \, \mathrm{d}\omega_i \;\; = \bar{L}_i \cos\bar{\theta} \tag{2}$$

where $\bar{L}_i$ is coming in from only *one single* direction $\bar{\omega} = (\bar{\theta}, \bar{\phi})$

- The outgoing irradiance becomes

$$M = \int_\Omega \int_\Omega \rho(\omega_o, \omega_i) \bar{L}_i \delta(\omega_i - \bar{\omega}) \cos\theta_i \cos\theta_o \, \mathrm{d}\omega_i \, \mathrm{d}\omega_o$$

$$= \int_\Omega \rho(\omega_o, \bar{\omega}) \bar{L}_i \cos\bar{\theta} \cos\theta_o \, \mathrm{d}\omega_o \tag{3}$$

- Plugging (2) and (3) into (1) gives:

$$\frac{M}{E} = \frac{\int_\Omega \rho(\omega_o, \bar{\omega}) \bar{L}_i \cos\bar{\theta} \cos\theta_o \, d\omega_o}{\bar{L}_i \cos\bar{\theta}} = \int_\Omega \rho(\omega_o, \bar{\omega}) \cos\theta_o \, d\omega_o \overset{!}{\leq} 1$$

- Remember that energy conservation holds for each and every L_i
- In total, we have the following necessary condition for BRDF's:

$$\forall \omega_i : \int_\Omega \rho(\omega_o, \omega_i) \cos\theta_o \, d\omega_o \overset{!}{\leq} 1$$

  - This will be important to check for concrete functions ρ !

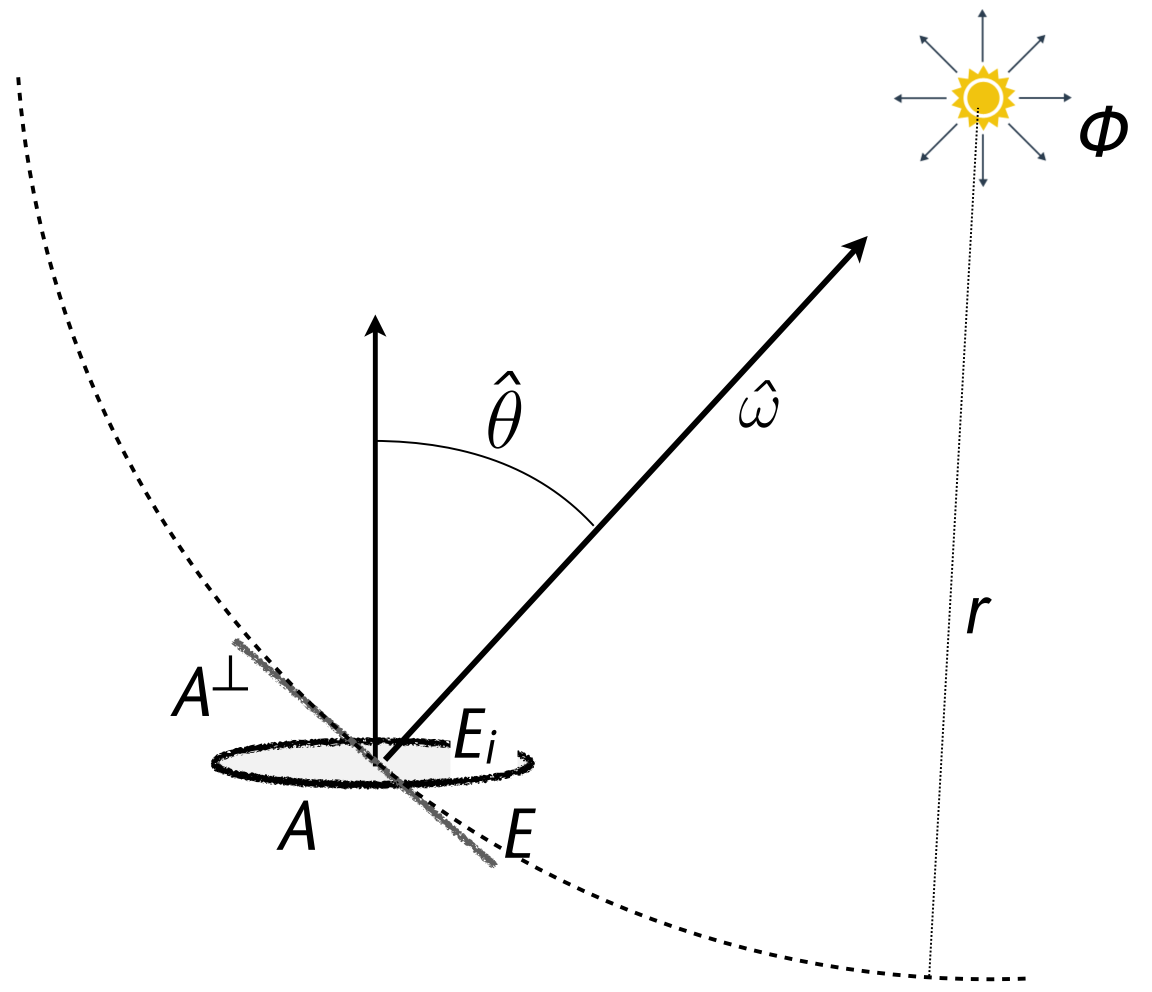
Practical Computation of L_o in Case of Illumination with a Point Light

- Note: in real life, point lights never occur (except for stars)
- Assumption: the point light shines in all directions isotropically, with flux Φ
- First: compute irradiance at A
  - Irradiance at $A^{\perp}$:

$$E = \frac{\Phi_i}{A^{\perp}} = \frac{\Phi}{4\pi r^2}$$

  - Irradiance on A (see slide 4):

$$E(\hat{\omega}) = \cos\hat{\theta} \cdot E = \cos\hat{\theta} \cdot \frac{\Phi}{4\pi r^2}$$



- Start with definition of radiance (slide 10):

$$L_i = \frac{\mathrm{d}^2\Phi}{\cos\theta_i \,\mathrm{d}A\,\mathrm{d}\omega_i}$$

- Resolve to get dE (slide 4):

$$\Rightarrow \quad L_i \cos\theta_i \,\mathrm{d}\omega_i = \frac{\mathrm{d}^2\Phi}{\mathrm{d}A} = \mathrm{d}E$$

- Use reflectance equation and insert previous eq.:

$$L_o = \int_\Omega \rho L_i \cos\theta_i \,\mathrm{d}\omega_i \;\; = \int_\Omega \rho \,\mathrm{d}E(\omega_i)$$

- Since it's a point light source, we receive light from exactly one direction $\hat{\omega}$
  - So, use the "trick" with the Dirac delta function again
- Outgoing radiance can be written as:

$$L_o = \int_\Omega \rho\delta(\omega_i - \hat{\omega})\,\mathrm{d}E(\omega_i) \quad = \rho E(\hat{\omega}) \quad \underset{\text{(slide 28)}}{=} \rho \frac{\cos\hat{\theta}}{4\pi r^2}\Phi$$

Constructing Actual BRDF's

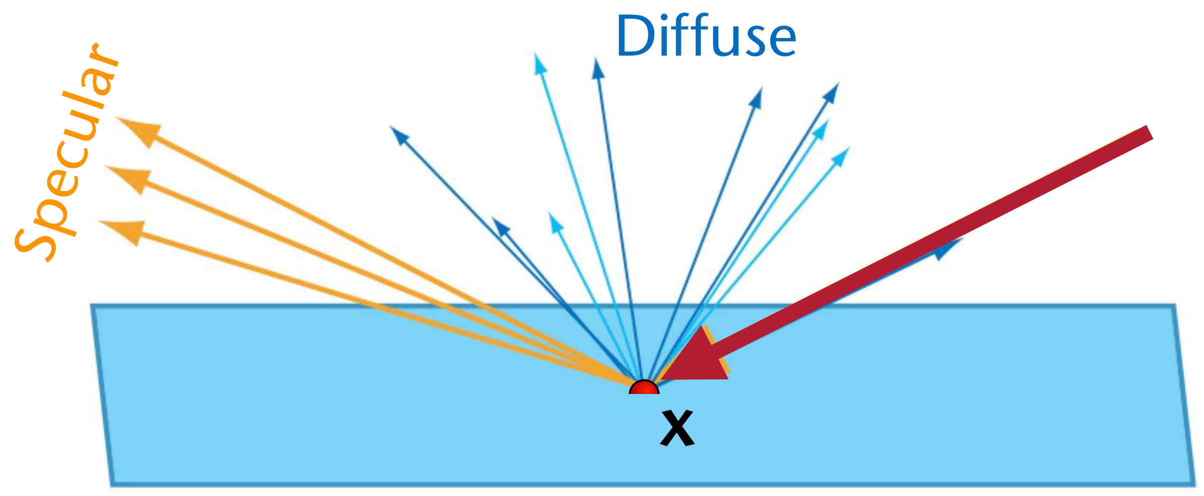
- Two types of reflections: diffuse and specular
- Accordingly, a BRDF consist of

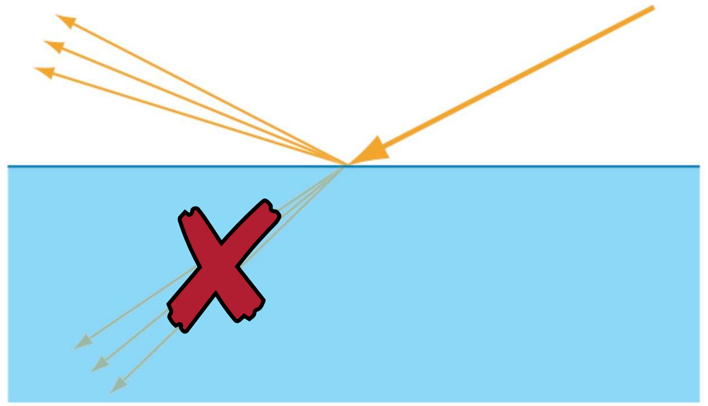
$$\rho = \rho_{\text{diff}} + \rho_{\text{spec}}$$

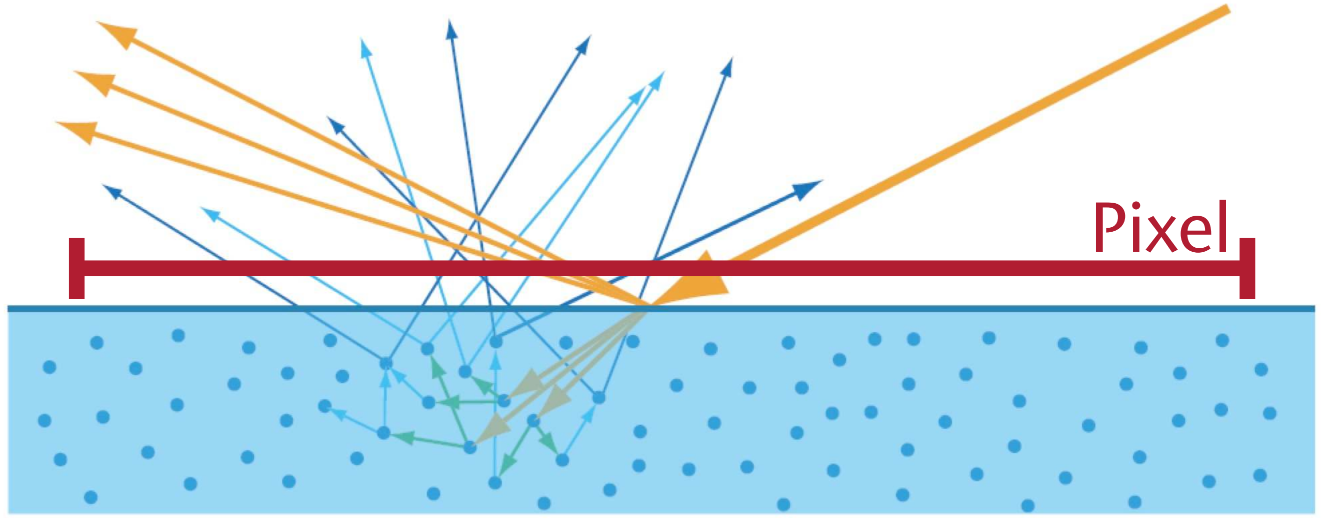
  - The diffuse term is simply the Lambert model:

$$\rho_{\text{diff}}(\boldsymbol{l}, \mathbf{v}) = \frac{1}{\pi} C_{\text{base}}$$

- Two types of materials: metallic and non-metallic (aka. dielectric)
  - Metallic: no diffuse term
  - Non-metallic: diffuse term due to internal scattering, but travel distance ≪ pixel size

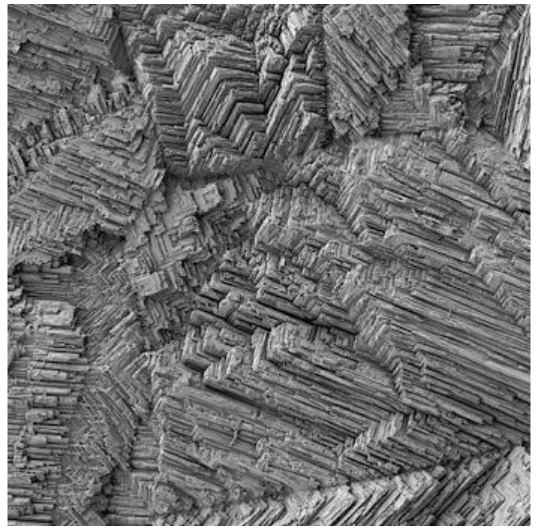


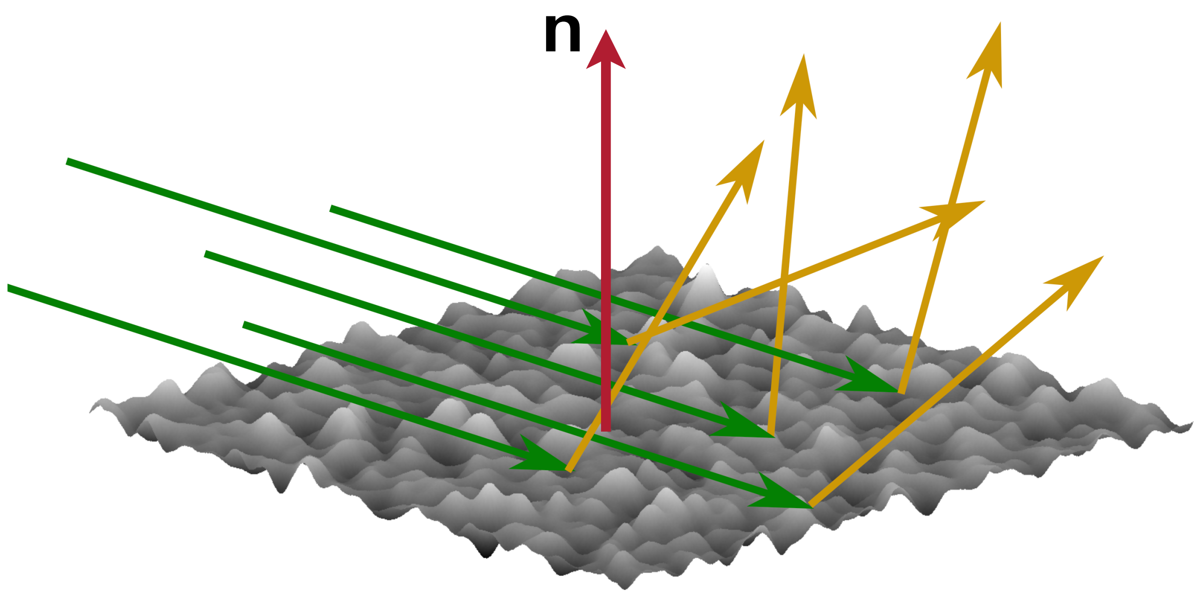




Constructing the Specular Term

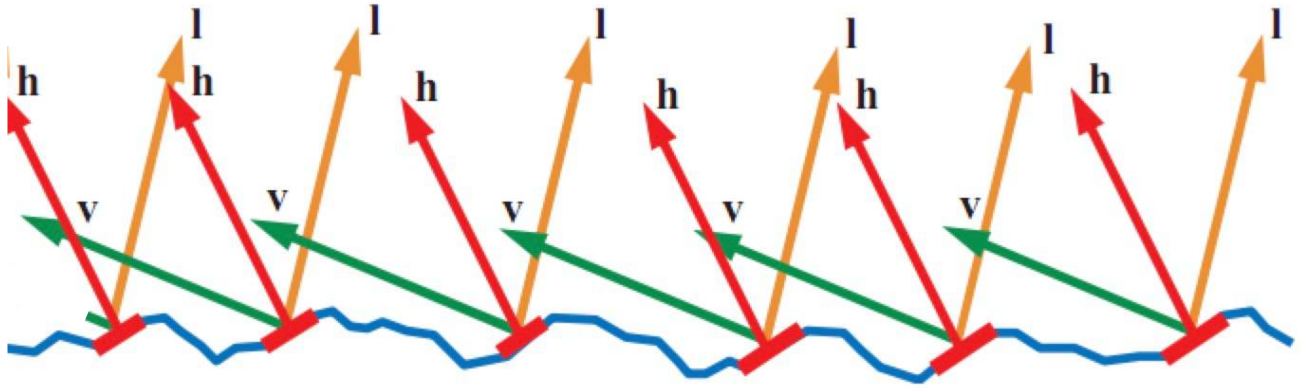
- Usually based on the micro-facet theory
- Even (apparently) smooth surfaces consist of microscopic facets
- Each micro-facet acts like a small perfect mirror
- Macroscopic surface normal **n** = "average" of all micro-facet normals, **m**
- Variance of all micro-facets around **n** depends on (or determines) the roughness of the surface



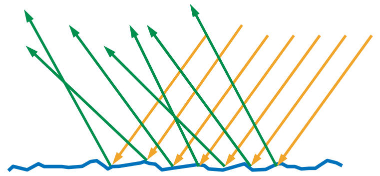


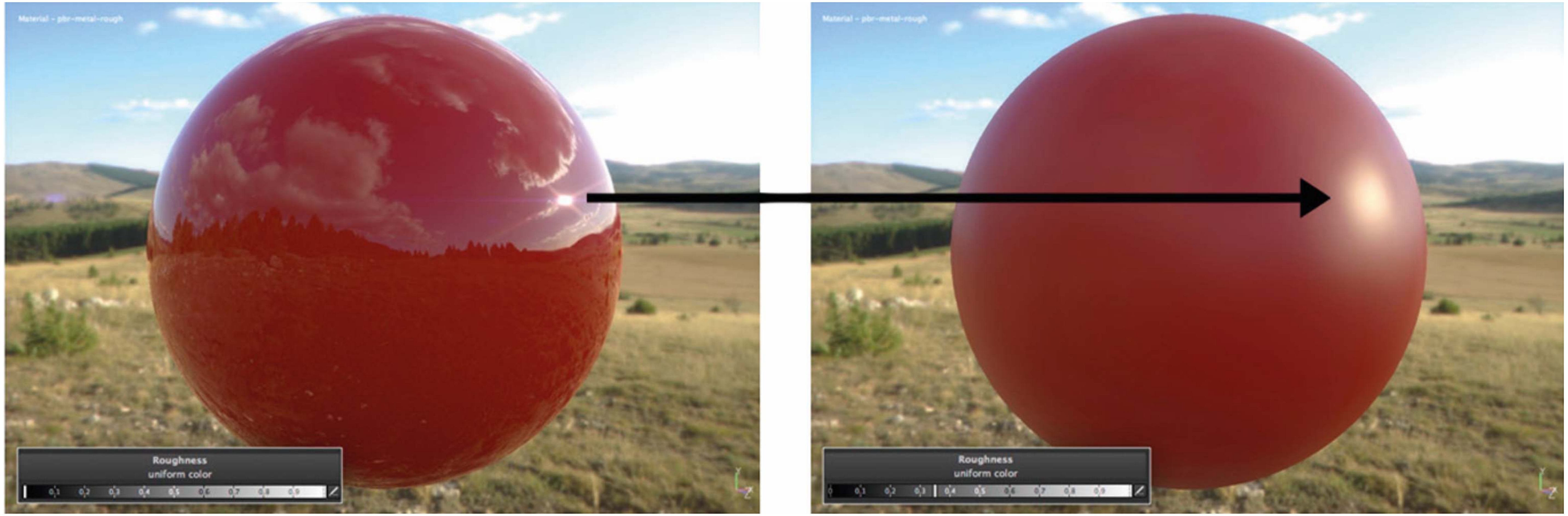
1. The Normal Distribution Function

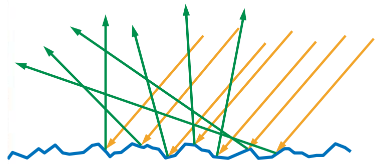
- For given $\boldsymbol{l}$ (ω_i) and **v** (ω_o), we are only interested in micro-facets with **m** = **h**



$\longrightarrow$ Normal distribution function (NDF), $D(\mathbf{h})$, describes "roughness"





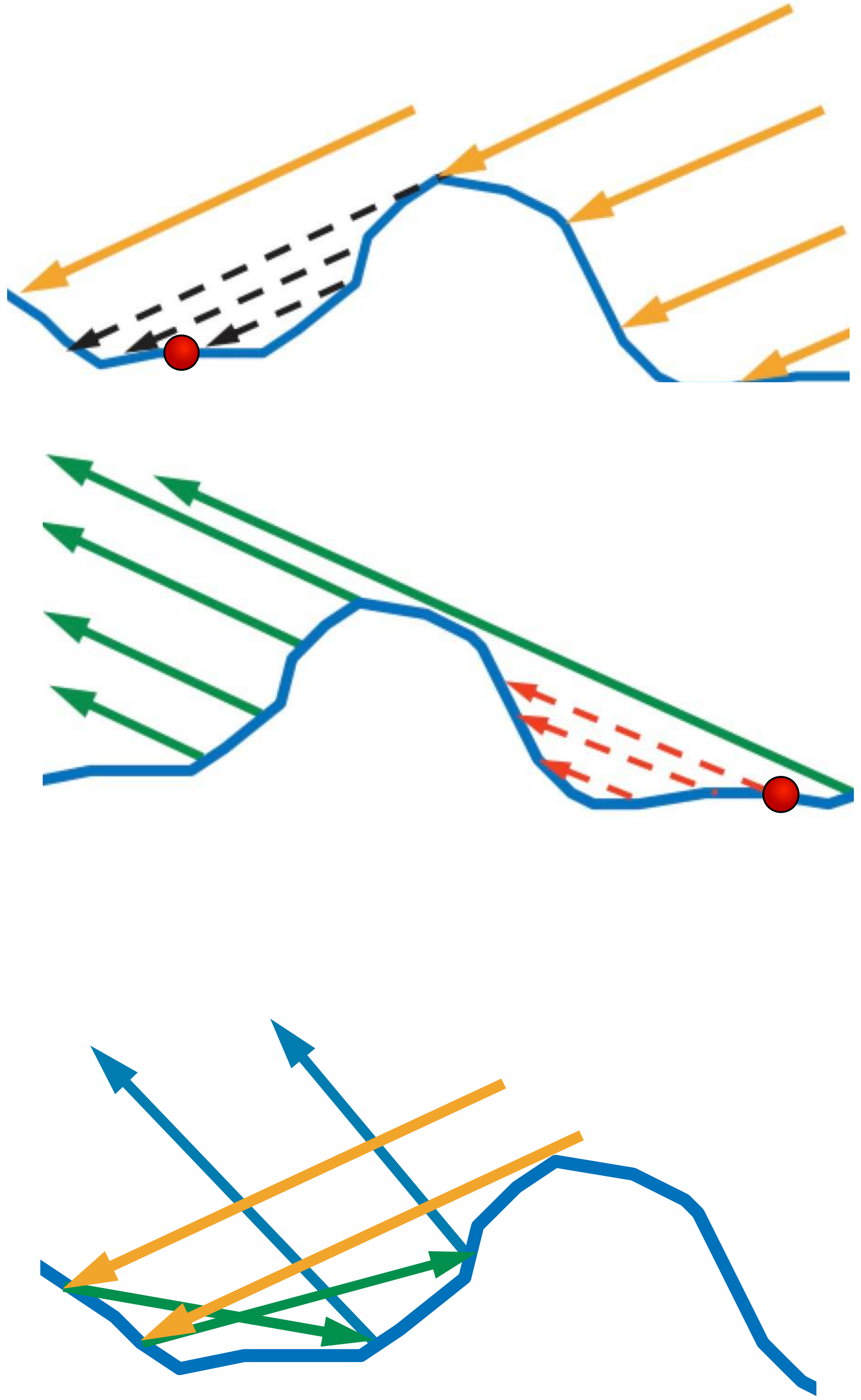


2. Geometry Function

- Shadowing and masking effects:
  - Other micro-facets, i.e. with $\mathbf{m} \neq \mathbf{h}$, can block either incoming light (shadowing) or are invisible from the outgoing direction (masking)
- Model these effects by a geometry function

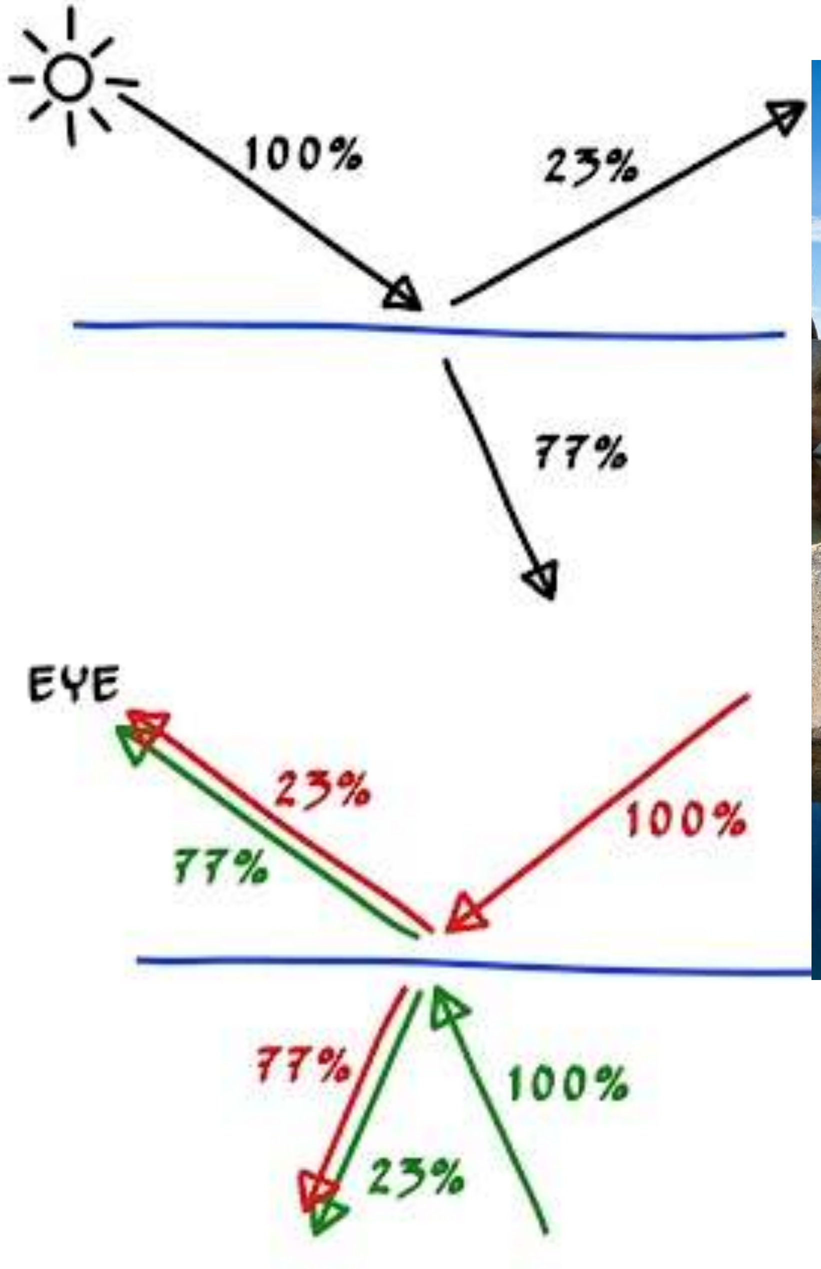
$$G(\boldsymbol{l}, \mathbf{v}, \mathbf{h})$$

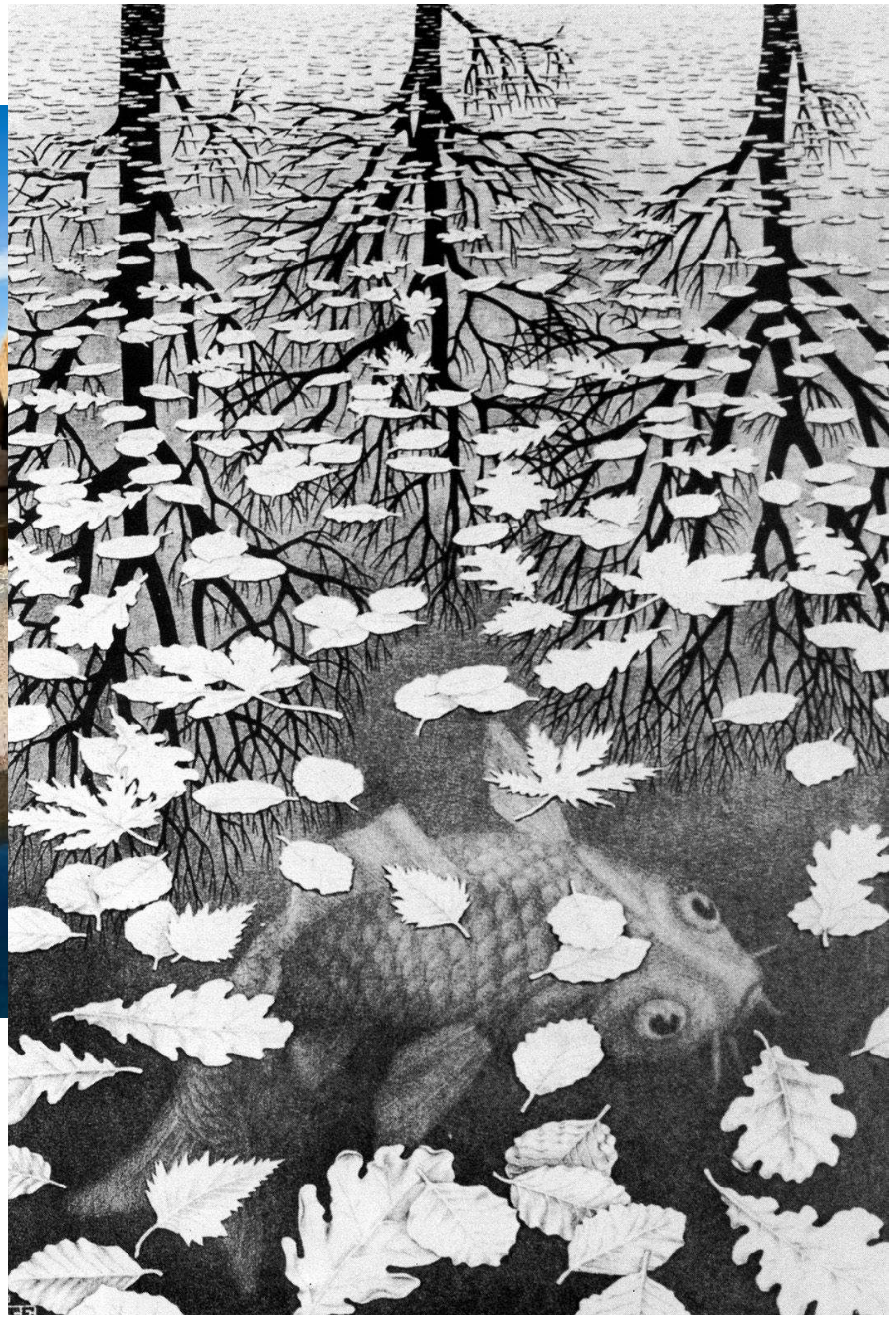
  - Tells the percentage of surface points with $\mathbf{m} = \mathbf{h}$ that are not shadowed or masked, as a function of the light direction $\boldsymbol{l}$ and the view direction $\mathbf{v}$
  - Inter-reflections are (usually) ignored



3. Fresnel Function

- Remember the Fresnel effect:
  - Some percentage of light (*F*) gets reflected, the rest (1-*F*) penetrates the surfaces
  - For surfaces between perfectly transparent materials, both incoming lights add nicely up to 100%
  - For opaque materials: the "rest" (1-*F*) gets absorbed
    - For semi-transparent materials: in-between; probably best left to the artist
- Modeled by the Fresnel function *F*(*θ*)





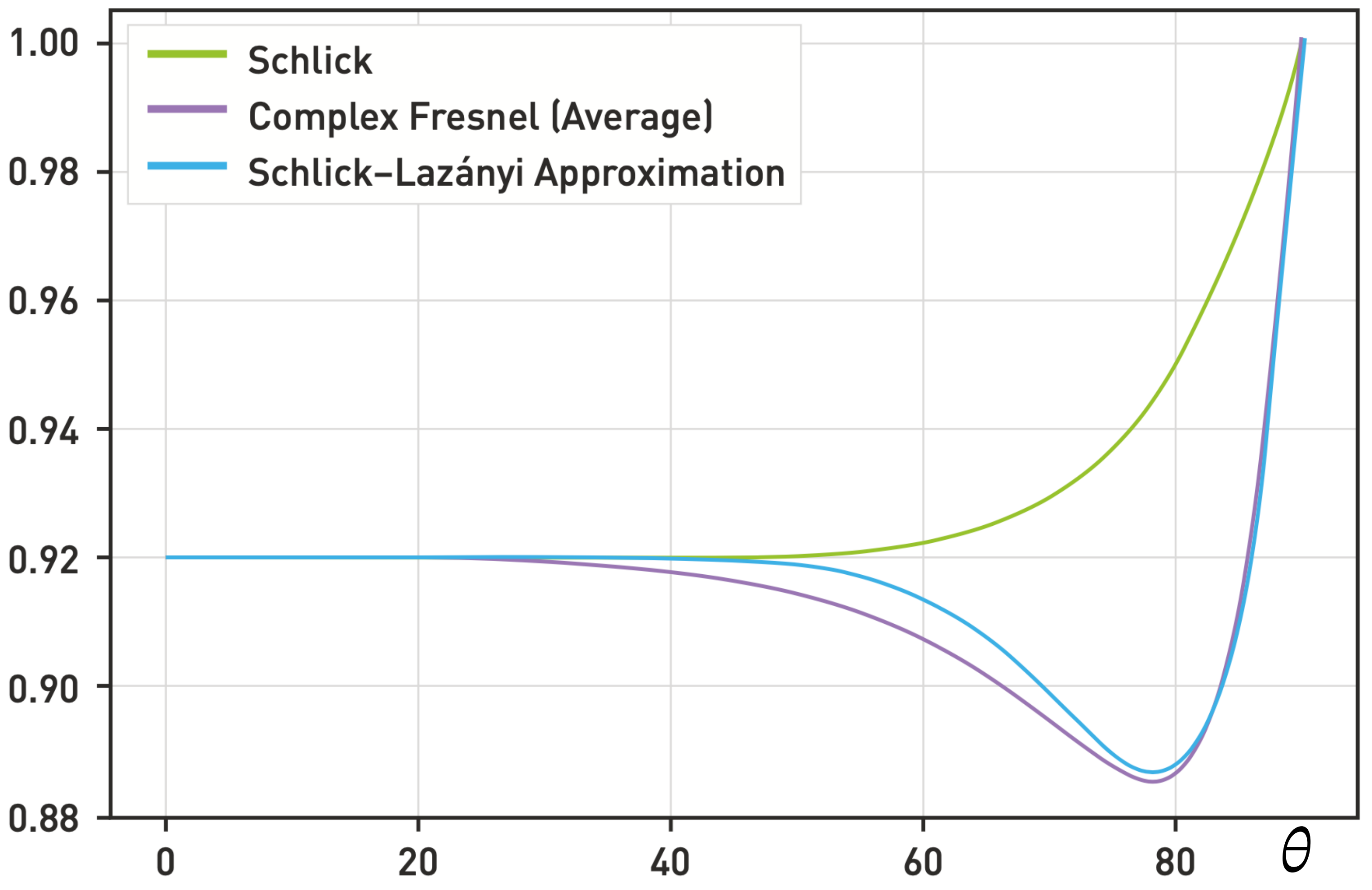
[M. C. Escher]

The Schlick-Lazányi Approximation For Metals

- For metals, a more complex approximation should be used:

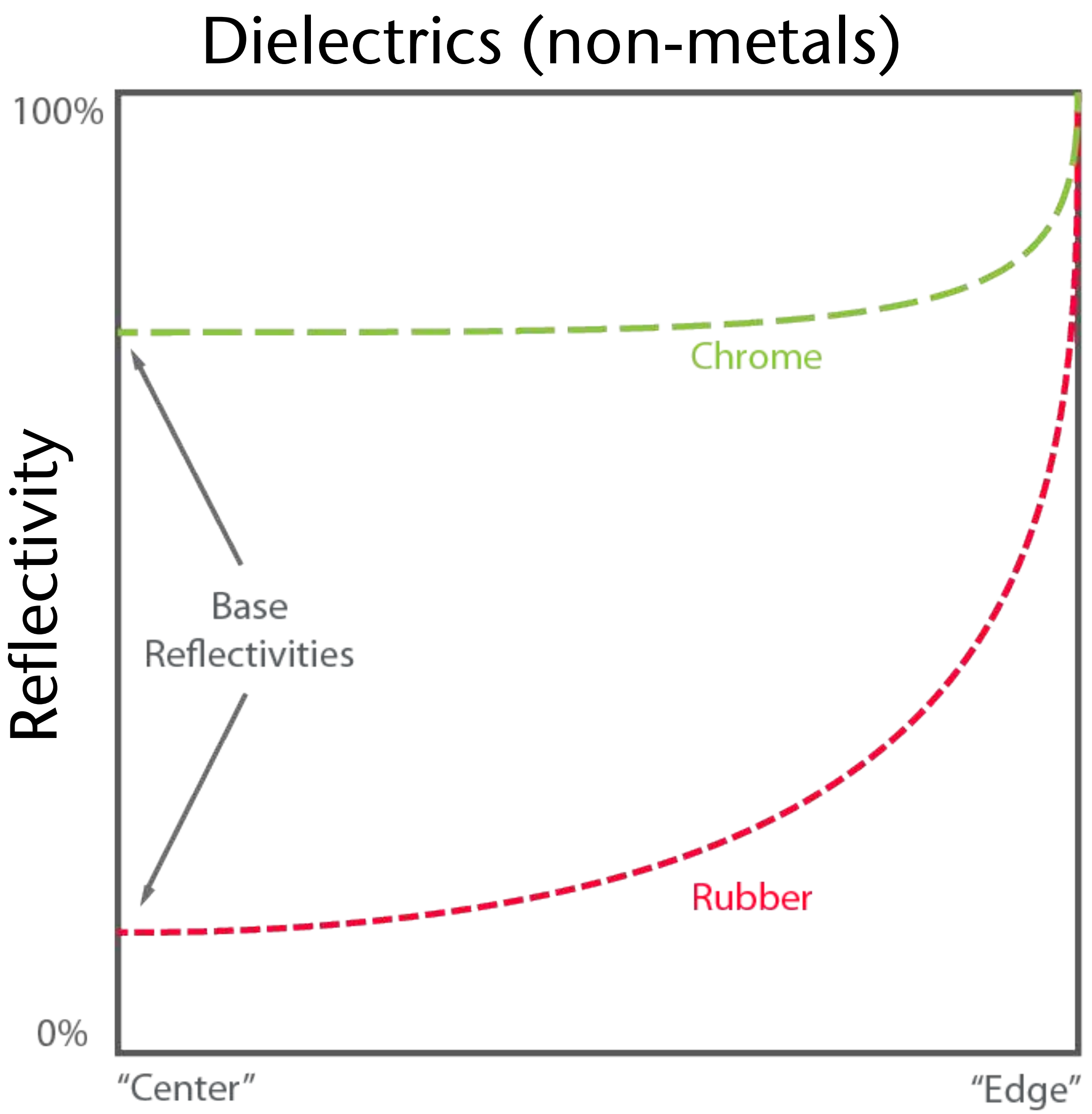
$$\rho(\theta) \approx \rho_0 + (1 - \rho_0)\,(1 - \cos\theta)^5 - a\cos\theta(1 - \cos\theta)^6$$

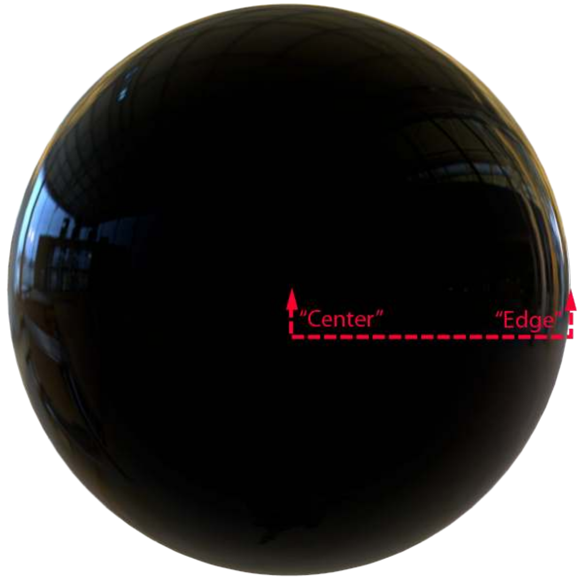
  - ρ = fraction of reflected light, rest of light gets absorbed
- Also, ρ_0 depends on the wavelength! (not so for dielectrics)
- Comparison:

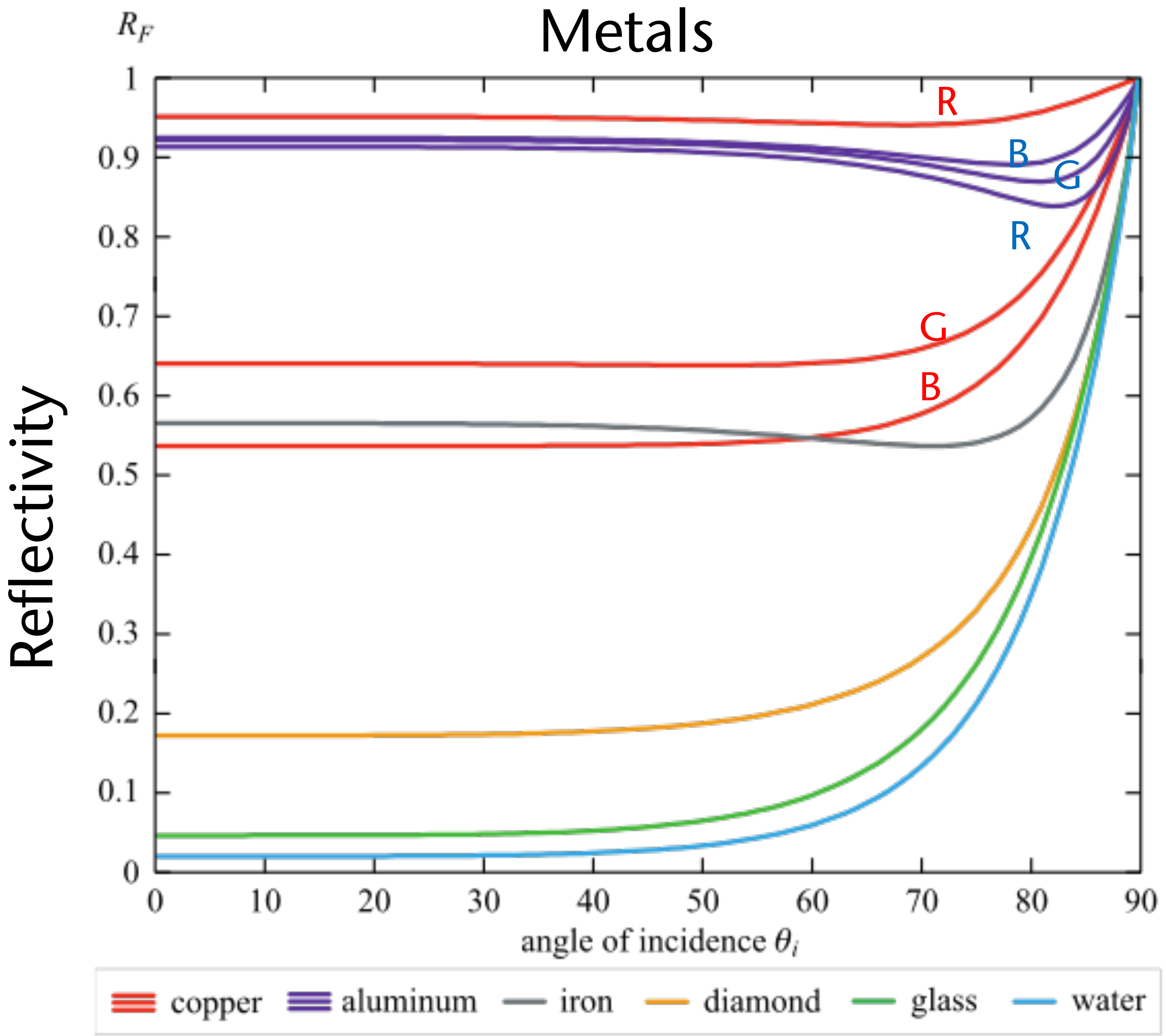


Aluminum,
incident light at 450 nm

Difference between Dielectrics and Metals







Notice: different curves for different wavelengths!

The Micro-Facet BRDF as a Whole

- Just multiply all three terms (and normalize, so the conditions are met)

$$\rho_{\text{spec}}(\boldsymbol{l}, \mathbf{v}) = \frac{F(\boldsymbol{l}, \mathbf{h}) D(\mathbf{h}) G(\boldsymbol{l}, \mathbf{v})}{4(\mathbf{n} \cdot \boldsymbol{l})(\mathbf{n} \cdot \mathbf{v})}$$

  - Note: evaluate only, if $\mathbf{n} \cdot \boldsymbol{l} \geq 0$ and $\mathbf{n} \cdot \mathbf{v} \geq 0$
    (otherwise, either light source, or outgoing direction is "below" surface)
  - Observe: when denominator approaches 0, so does G
    - Just check the definition of G
- Note: there are a lot of variants, this is just the bare concept

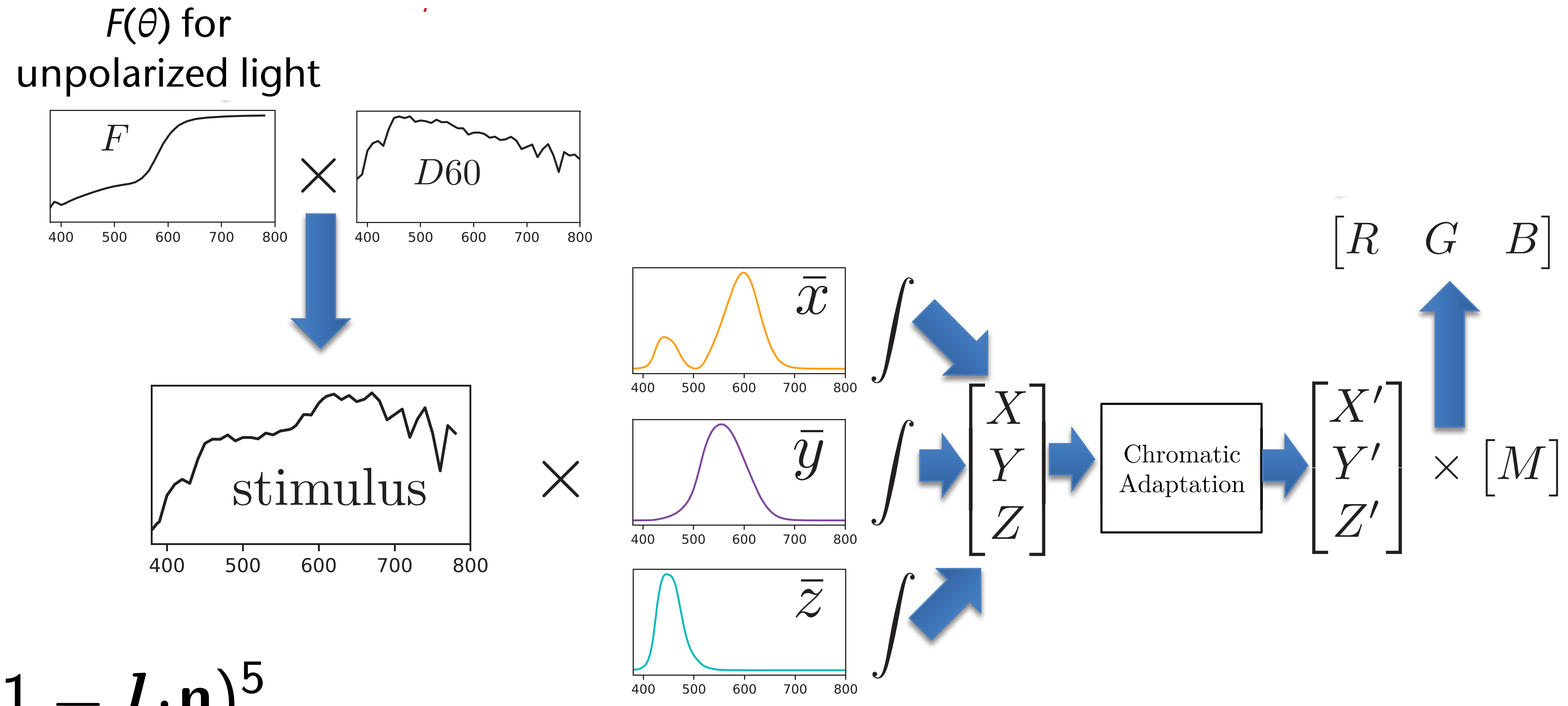
Approximation of the Fresnel Equation

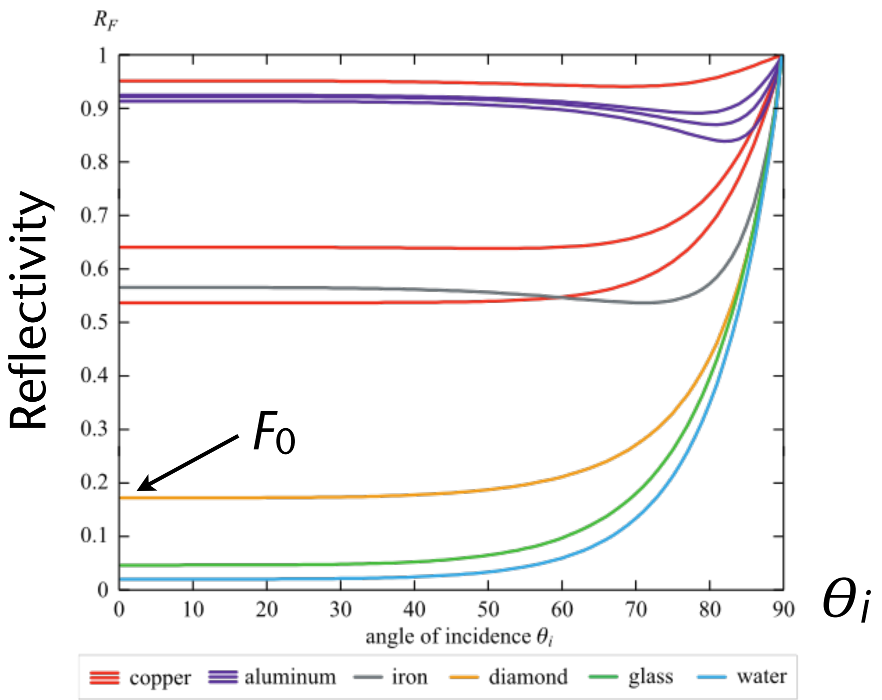
- Physically correct model is extremely complex
  - Depends on wavelength
  - Depends on polarization
- Good approximation is the one by Schlick:

$$F(\boldsymbol{l}, \mathbf{n}) = F_0 + (1 - F_0)(1 - \boldsymbol{l}\cdot\mathbf{n})^5$$

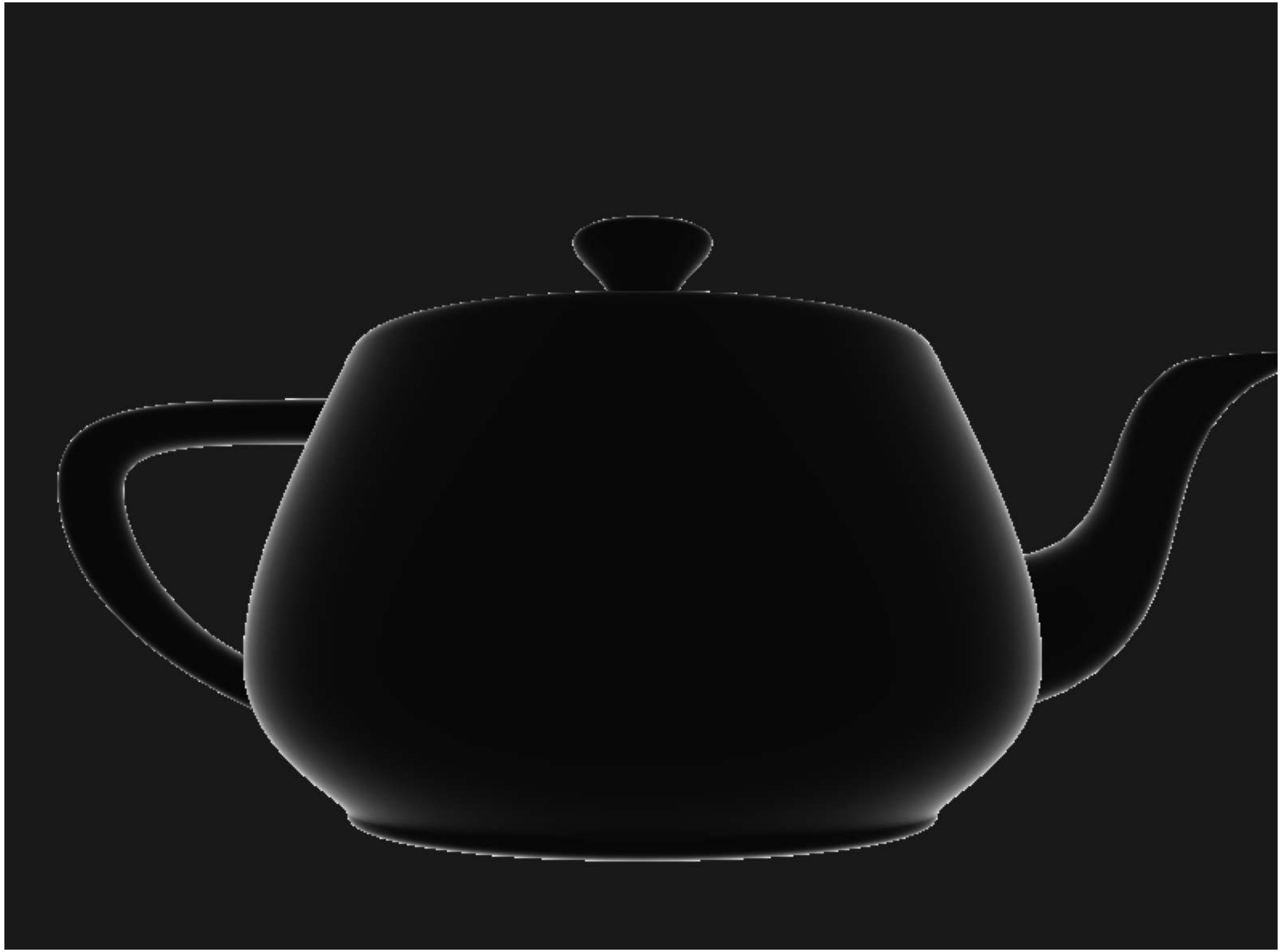
- In micro-facet models: only micro-facets with $\mathbf{m} = \mathbf{n}$ and $\mathbf{m} = \mathbf{h} = \frac{1}{2}(\boldsymbol{l} + \mathbf{v})$
- So, in the micro-facet BRDF:

$$F(\boldsymbol{l}, \mathbf{h}) = F_0 + (1 - F_0)(1 - \boldsymbol{l}\cdot\mathbf{h})^5$$





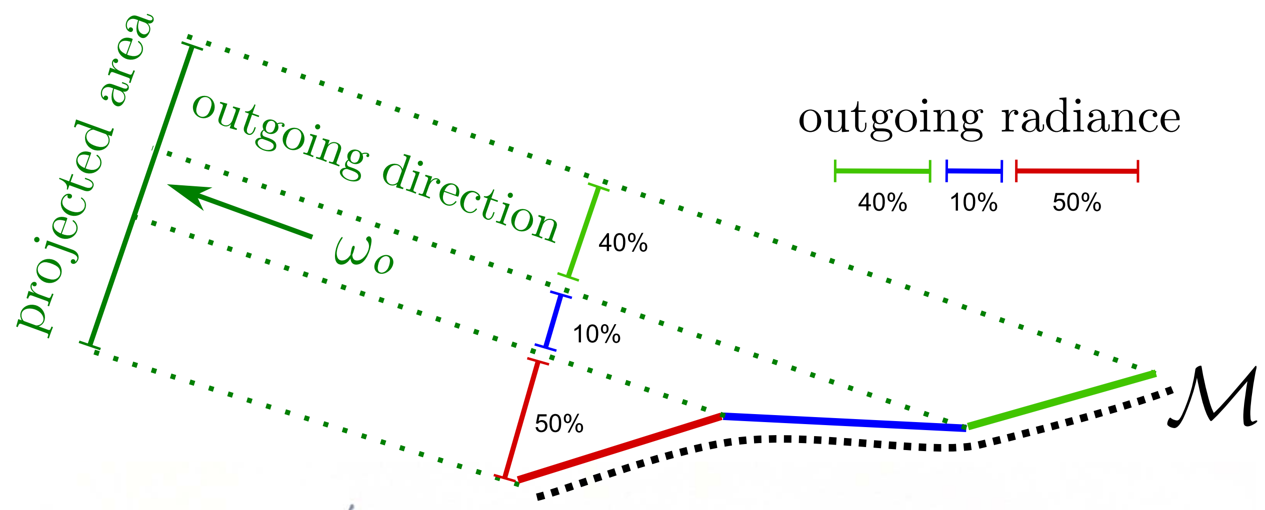
Visualization of the Fresnel Term

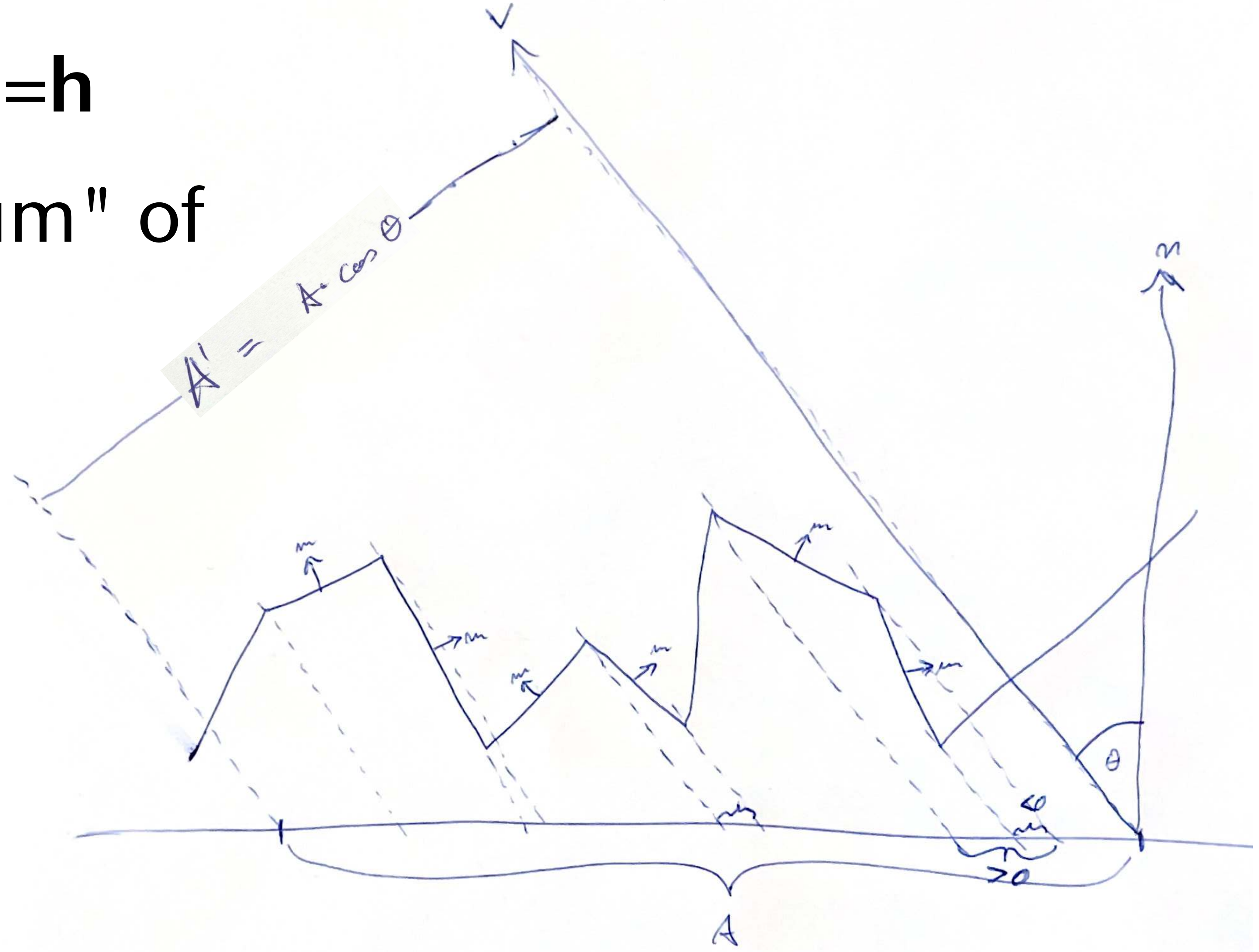


The Normal Distribution Function in Detail

- Roughness is the parameter in any NDF, *D*, determining the variance of the micro-normals around the macroscopic normal
- Intuitive interpretation: $D(\mathbf{h})$ = percentage of the micro-surface $\mathcal{M}$ that has micro-normals **m=h**
- Important property: for any NDF *D*, the "sum" of the area of the micro-facets, when projected on outgoing direction **v**, must equal the area of the macro-surface, projected on **v**

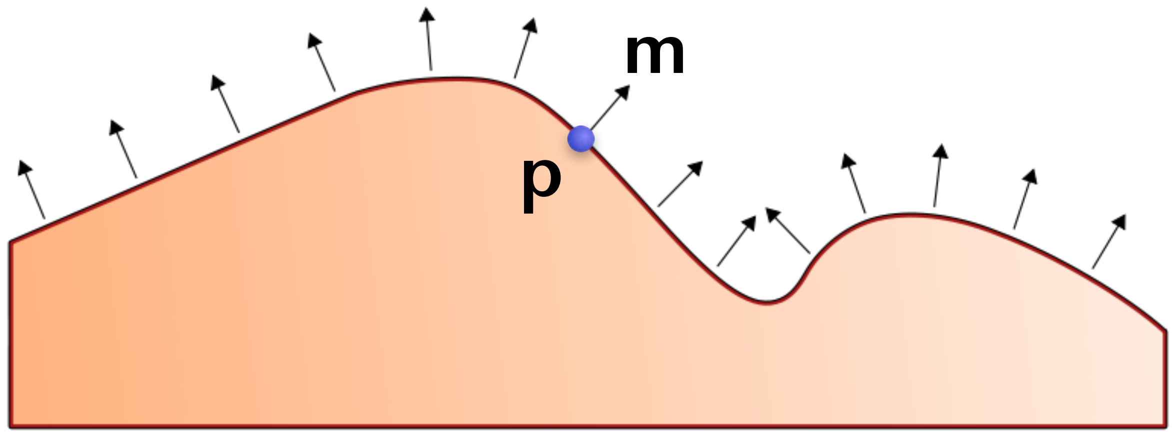
$$(\mathbf{v}\cdot\mathbf{n}) = \int_{\Omega} D(\mathbf{m}_\omega)(\mathbf{v}\cdot\mathbf{m})\, d\omega$$





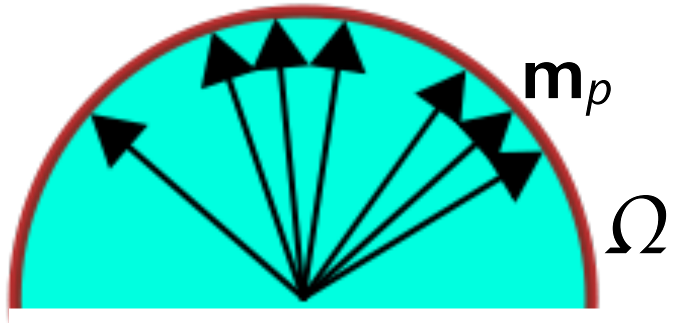
Another Important Property

- The NDF allows us to switch from spatial integrals over the surface to "statistical" integrals over the hemisphere of normals Ω



- The Gauß map: maps every point on a surface to its normal

$$G : \mathbf{p} \in M \mapsto \mathbf{m}_p \in \Omega$$



- Consider $\Omega' \subseteq \Omega$; this induces

$$M' \subseteq M \quad \text{with} \quad M' = \{\mathbf{p} \in M \,|\, \mathbf{m}(\mathbf{p}) \in \Omega'\}$$

- The NDF D has (must have) the property (sketch of proof later):

$$\int_{M'} \mathrm{d}p = \int_{\Omega'} D(\mathbf{m}) \, \mathrm{d}m \qquad (*)$$

Nice consequences

- The area can be computed two ways:

$$\text{Area}(M) = \int_M \mathrm{d}p = \int_\Omega D(\mathbf{m})\,\mathrm{d}m$$

- Integrals over the surface can be converted into integrals over the hemisphere:

$$\int_M f(\mathbf{m}(\mathbf{p}))\,\mathrm{d}p = \int_\Omega f(\mathbf{m})D(\mathbf{m})\,\mathrm{d}m$$

FYI: Sketch of Proof for Equation (*)

- Let M be the micro-surface, $\mathbf{p}$ a point on M
- From differential geometry:

$$\kappa(\mathbf{p}) = \lim_{A \to 0} \frac{A'}{A}$$

where A = area of small patch around $\mathbf{p} \in M$, shrinking smaller and smaller,

$A' = G(A) = \{\mathbf{m}(\mathbf{p}) \,|\, \mathbf{p} \in A\} \subseteq \Omega$ (Gauß map of A) is a small area on Ω

- For small A, we can also write $\kappa(\mathbf{p}) = \frac{\mathrm{d}A'}{\mathrm{d}A}$, or $\mathrm{d}A = \frac{1}{\kappa(\mathbf{p})} \mathrm{d}A'$
- Replacing $\mathrm{d}A'$ (area on Ω) with $\mathrm{d}\omega_{\mathbf{p}}$ (solid angle), we can replace integrals over micro surfaces by integrals over Ω:

$$\int_M f(\mathbf{m}(\mathbf{p})) \, \mathrm{d}\mathbf{p} \, \mathrm{d}A = \int_\Omega f(\mathbf{m}) \frac{1}{\kappa(\mathbf{p})} \, \mathrm{d}\mathbf{m} \, \mathrm{d}\omega$$

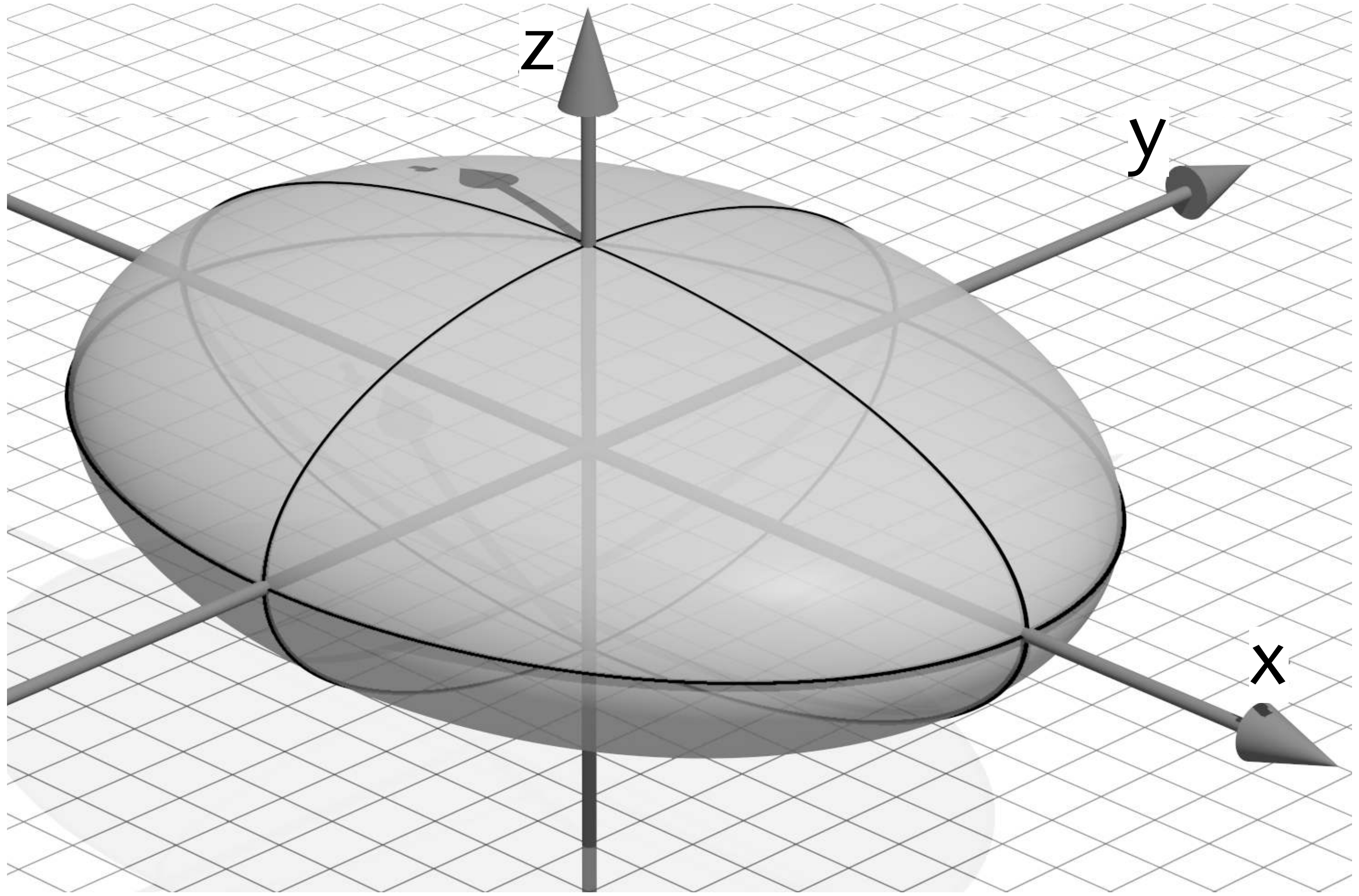
Derivation of an NDF

- Assumption: normals on M are distributed like on an ellipsoid
- General (implicit) equation of general ellipsoids:

$$f(\mathbf{p}) = \mathbf{p}^\mathsf{T} A^\mathsf{T} A \mathbf{p} - 1 = 0$$

with $A = R \cdot S$, where R = rotation matrix, S = scaling

- Simplification: choose $A = \begin{pmatrix} \alpha_x & & \\ & \alpha_y & \\ & & 1 \end{pmatrix}$
  - α_x and α_y are kind of "roughness" params
- Note: the following is only a sketch of a derivation - lots of steps/theorems in between, which are rather deep results from differential geometry, have been omitted (otherwise, much more time would be needed in class)

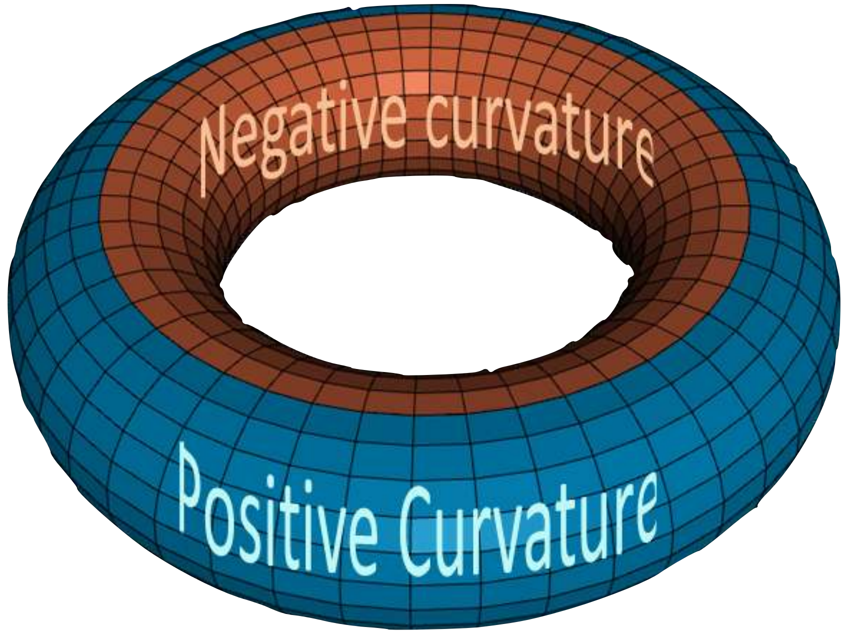


- From differential geometry: integrals over surfaces can be translated into integrals over the Gauss map, using the factor

$$D(\mathbf{m}) = \frac{1}{\ldots \kappa(\mathbf{m})}$$

  where κ = Gaussian curvature

  (The "..." are normalization terms to maintain conservation of energy, omitted here for clarity)



- Next goal: derive κ for our special surface (ellipsoid)

- More from differential geometry: implicit surfaces have the following Gaussian curvature at point **p** on the surface

$$\kappa(\mathbf{p}) = \frac{(\nabla f)^{\mathsf{T}} \mathrm{adj}(H) \nabla f}{||\nabla f||^4} \tag{1}$$

where

$$\nabla f = \text{derivative of } f = 2A^2\mathbf{p} = H\mathbf{p}$$

$$H = \nabla^2 f = \text{Hessian matrix of } f = 2A^2$$

$$\mathrm{adj}(H) = \text{adjoint matrix of } H = \det(H) H^{-1} = 4 \det(A)^2 A^{-2}$$

- For all implicit surfaces, the normal **m** at a point **p** on the surface is

$$\mathbf{m} = \frac{\nabla f(\mathbf{p})}{||\nabla f||}$$

- Plugging everything into (1):

$$\kappa(\mathbf{p}) = \frac{\det(H)\mathbf{m}^{\mathsf{T}} H^{-1}\mathbf{m}}{||\nabla f||^2} \tag{2}$$

- Remember: *f* is an ellipsoid, so there must be a 1:1 mapping between normals **m** (on the Gauss map) and points **p** (on the surface)
- Next goal: get rid of the $||\nabla f||^2$ in the denominator

- Look at the term $\mathbf{m}^\mathsf{T} H^{-1} \mathbf{m}$
- Use $\mathbf{m} = \frac{\nabla f}{||\nabla f||}$, plugging it into previous term, we get

$$\mathbf{m}^\mathsf{T} H^{-1} \mathbf{m} = \frac{(\nabla f)^\mathsf{T} \cdot H^{-1} \cdot \nabla f}{||\nabla f||^2} = \frac{(H\mathbf{p})^\mathsf{T} \cdot H^{-1} \cdot (H\mathbf{p})}{||\nabla f||^2} = \frac{\mathbf{p}^\mathsf{T} H^\mathsf{T} \mathbf{p}}{||\nabla f||^2}$$

$$= \frac{2\mathbf{p}^\mathsf{T} (A^2)^\mathsf{T} \mathbf{p}}{4||A^2\mathbf{p}||^2} = \frac{1}{2||A^2\mathbf{p}||^2}$$

- Thus,

$$\frac{1}{||A^2\mathbf{p}||^2} = 2\mathbf{m}^\mathsf{T} H^{-1} \mathbf{m} = \mathbf{m}^\mathsf{T} A^{-2} \mathbf{m} \qquad (3)$$

- Now, we can rewrite (1):

$$\kappa(\mathbf{p}) = \frac{\det(H)\mathbf{m}^\mathsf{T} H^{-1}\mathbf{m}}{||\nabla f||^2} = \frac{2^3 \det(A)^2 \mathbf{m}^\mathsf{T} \frac{1}{2} A^{-2}\mathbf{m}}{2^2 ||A^2\mathbf{p}||^2} = \det(A)^2(\mathbf{m}^\mathsf{T} A^{-2}\mathbf{m})(\mathbf{m}^\mathsf{T} A^{-2}\mathbf{m})$$

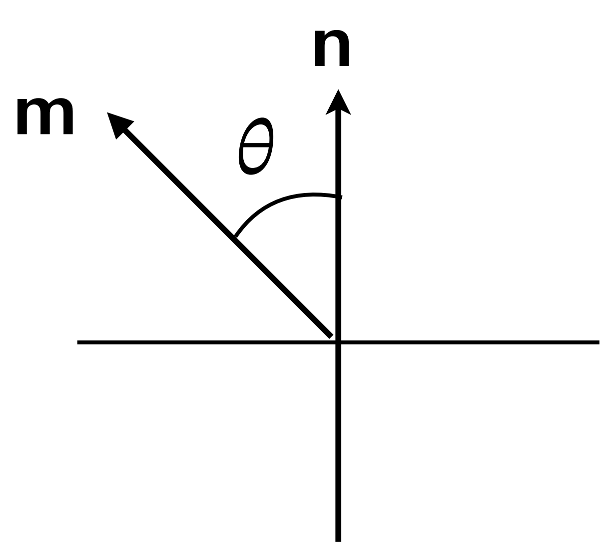
- We can simplify this further by using $\mathbf{m}^\mathsf{T} A^{-2}\mathbf{m} = (A^{-1}\mathbf{m})^2$
  - This is because A is special: $\mathbf{m}^\mathsf{T} A^{-2}\mathbf{m} = \mathbf{m}^\mathsf{T}(A^{-1})^\mathsf{T} A^{-1}\mathbf{m} = (A^{-1}\mathbf{m})^\mathsf{T} A^{-1}\mathbf{m}$
- In total: $\kappa(\mathbf{m}) = \det(A)^2(A^{-1}\mathbf{m})^4$
- Thus: $D(\mathbf{m}) = \frac{1}{\kappa(\mathbf{m})} = \frac{1}{(\alpha'_x\alpha'_y)^2(A'\mathbf{m})^4}$ with $\alpha'_x = \frac{1}{\alpha_x}$, $A' = A^{-1} = \mathrm{diag}(\alpha'_x, \alpha'_y, 1)$

- Generalization: allow $A = R \cdot S$, with non-trivial rotation
  - Effect: predominant normals direction is *not* along macroscopic normal
  - Is it really necessary? would it be visible? are there such materials?
- Further specialization: choose $\alpha_x = \alpha_y = \alpha$ (i.e., isotropic NDF) (note: here, I will write α, instead of α', just for sake of simplicity)
  - Thus, we get

$$D(\mathbf{m}) = \frac{1}{\alpha^4 (A\mathbf{m})^4} \quad \text{with} \quad A = \text{diag}(\alpha, \alpha, 1)$$

  - You can use this equation as-is, or make it more obviously depend on θ

- Use polar coords to represent **m**: $\mathbf{m} = \begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi \sin\theta \\ \cos\theta \end{pmatrix}$



- Consider the term

$$(A\mathbf{m})^2 = \alpha^2(\cos\varphi\sin\theta)^2 + \alpha^2(\sin\varphi\sin\theta)^2 + (\cos\theta)^2$$

$$= \alpha^2(1-\cos^2\theta) + \cos^2\theta$$

(using twice $\cos^2+\sin^2=1$)

$$= \alpha^2 + (1-\alpha^2)\cos^2\theta \quad = \alpha^2\left(1 + (\frac{1}{\alpha^2} - 1)\cos^2\theta\right)$$

(just a bit of terms rearrangement)

- Remember, α is in reality $\frac{1}{\alpha}$, and $\cos\theta = \mathbf{n}\cdot\mathbf{m}$, so we stick both back in:

$$(A\mathbf{m})^2 = \frac{1}{\alpha^2}\left(1 + (\alpha^2 - 1)(\mathbf{nm})^2\right)$$

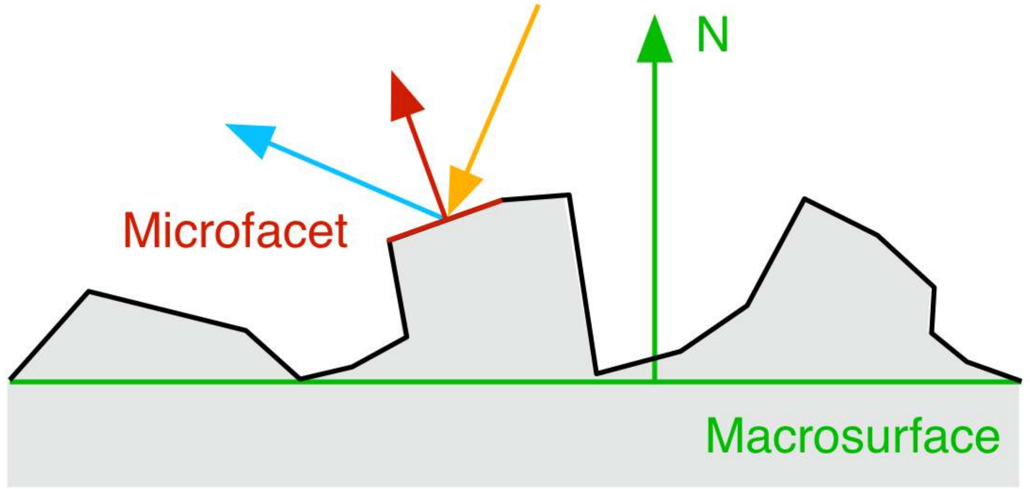
- In total, we get: $D(\mathbf{m}) = \dfrac{1}{\frac{1}{\alpha^4}(A'\mathbf{m})^4} = \dfrac{\alpha^8}{\left(1 + (\alpha^2-1)(\mathbf{nm})^2\right)^2}$

A Commonly Used NDF

- Often, the so-called Trowbridge-Reitz/GGX is used (which is isotropic wrt. roughness):

$$D(\mathbf{h}) = \frac{\alpha^2}{\pi((\mathbf{n}\cdot\mathbf{h})^2(\alpha^2-1)+1)^2}$$

  where α = roughness

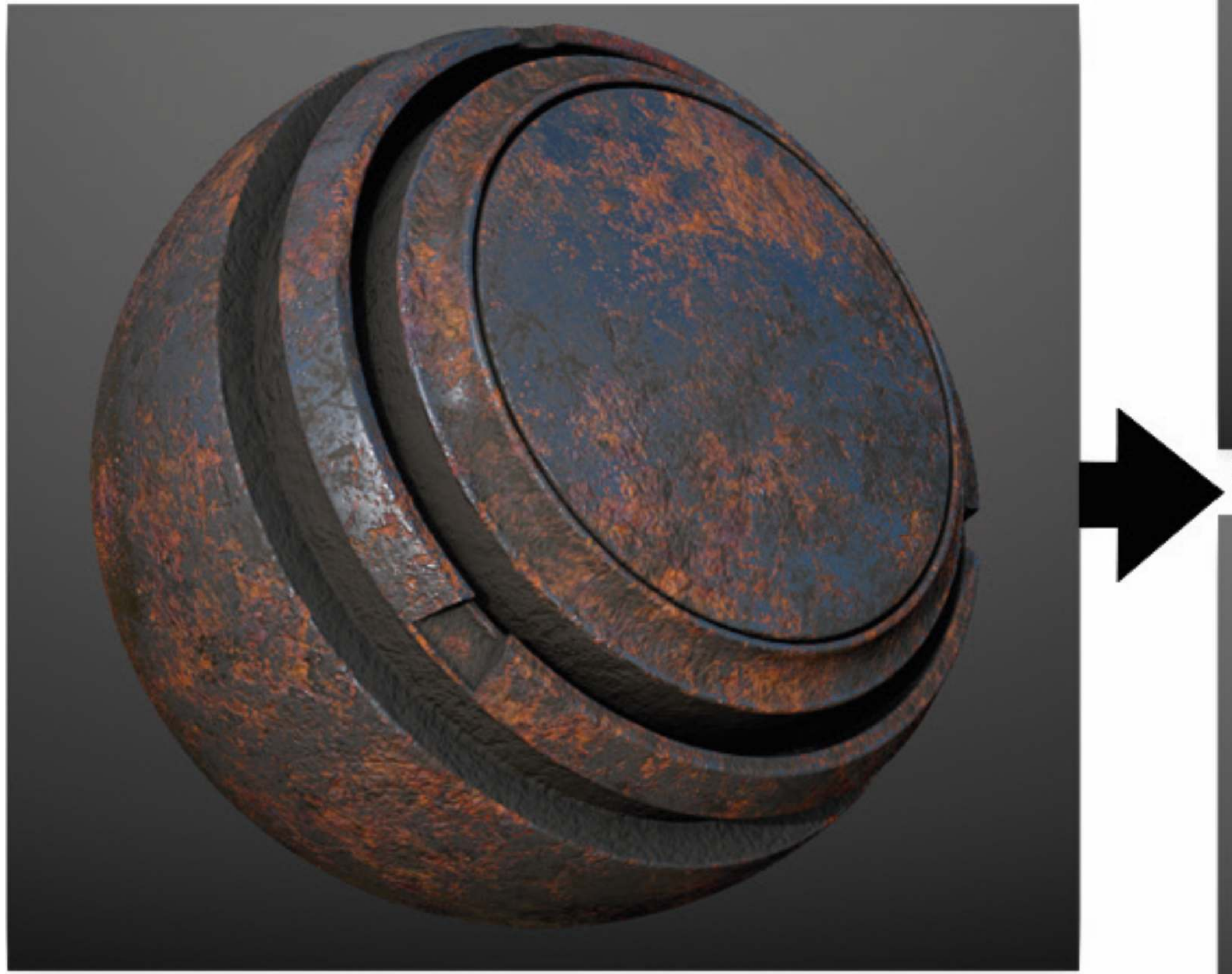


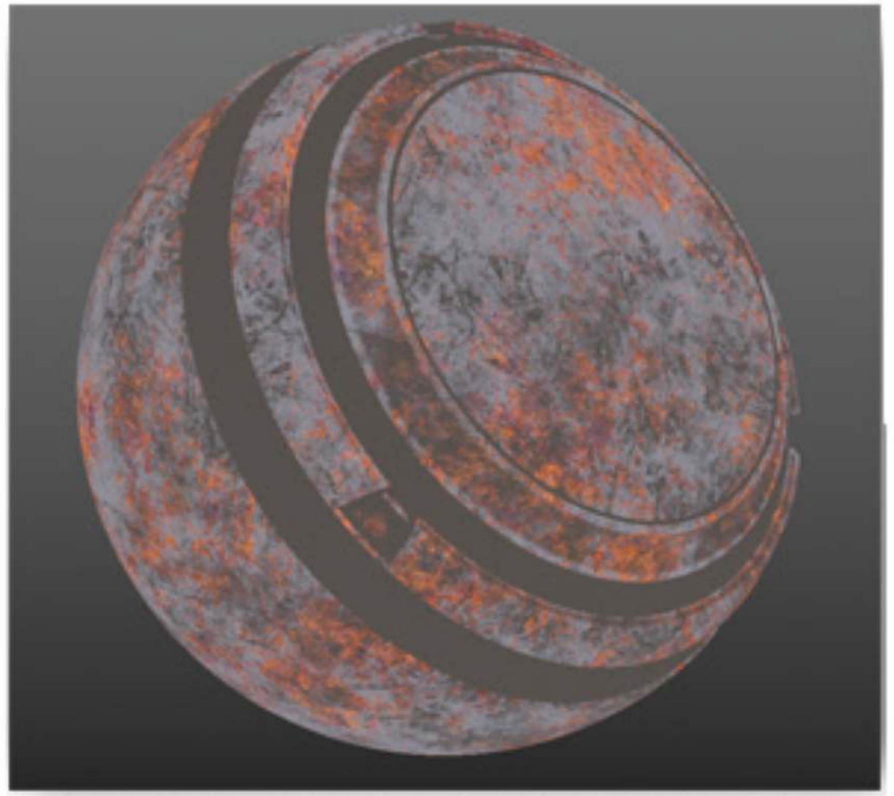
- Often, artists prefer the following transfer function:

$$\alpha = (1 - \text{glossiness})^2$$

  - Glossiness is more intuitive (brighter texels in the "roughness" texture means "more shiny")
  - Apparently, the square function is aesthetically more pleasing

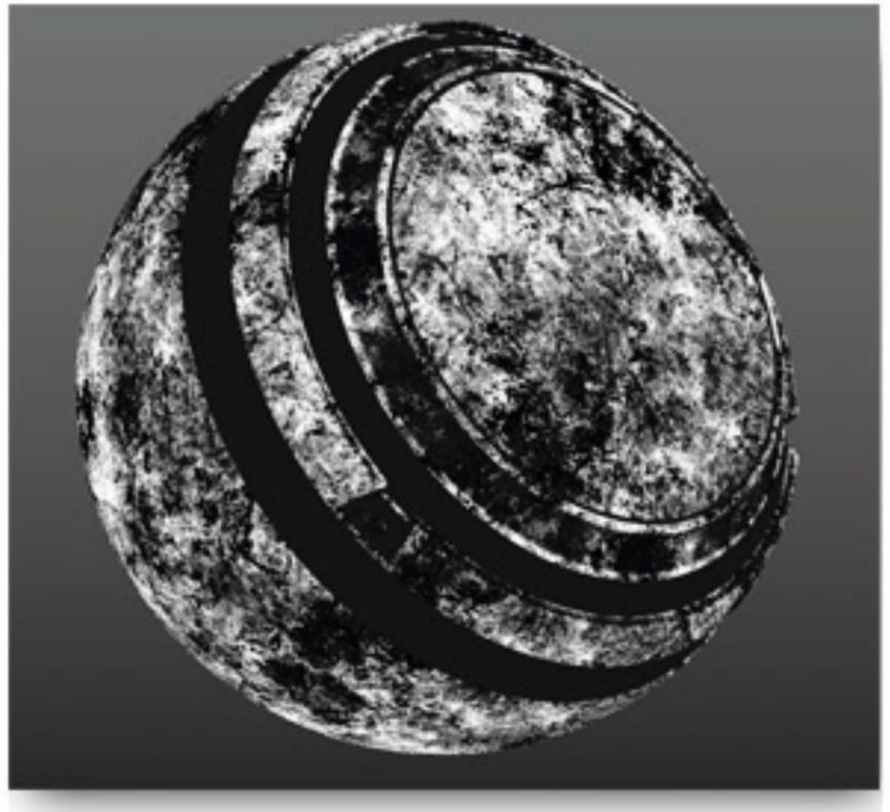
Example of Roughness, Color, and Metallic Textures





base color - RGB - interpreted as sRGB

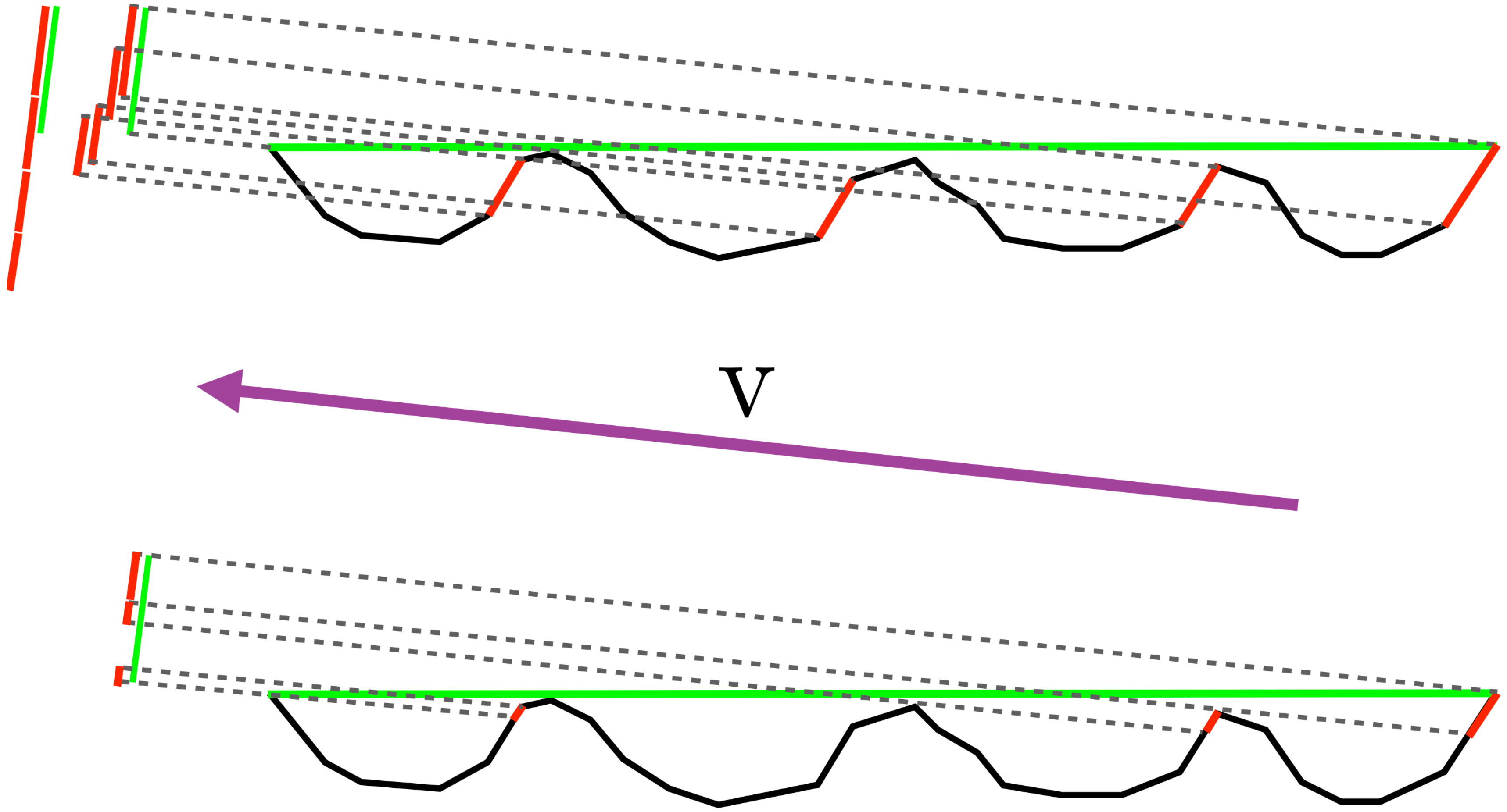
roughness - Grayscale - interpreted as linear



metallic- Grayscale - interpreted as linear

The Geometry Function

- The geometry function $G = G(\boldsymbol{l}, \mathbf{v}, \mathbf{h})$ gives the "likelihood" that micro-facets with $\mathbf{m} = \mathbf{h} = \frac{1}{2}(\boldsymbol{l} + \mathbf{v})$ are visible from the outgoing direction as well as for the incoming light



- Simplifications:
  - G is symmetric in $\boldsymbol{l}$ and $\mathbf{v}$
  - Whether or not a micro-facet is occluded does not depend on its orientation
- Commonly used function (Smith):

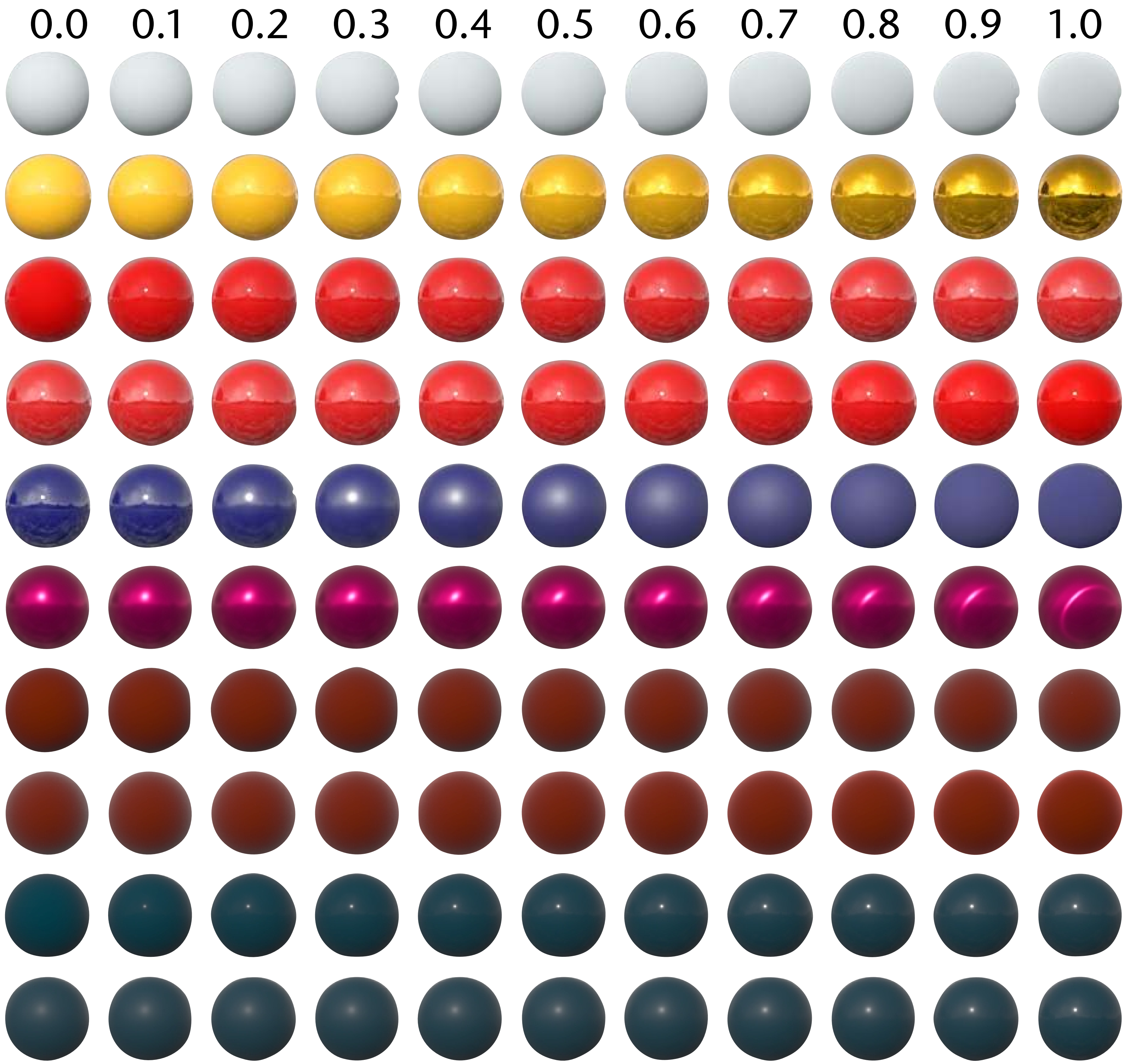
$$G(\boldsymbol{l}, \mathbf{v}, \mathbf{h}) = G'(\boldsymbol{l})G'(\mathbf{v}) \qquad G'(\mathbf{x}) = \frac{\mathbf{n} \cdot \mathbf{x}}{(\mathbf{n} \cdot \mathbf{x})(1 - \beta) + \beta}$$

where $\beta = \left(\frac{\text{roughness}+1}{2}\right)^2$ (again for aesthetic reasons)

A Current Standard is the "Disney BRDF"

- Sometimes in general called *metal/roughness workflow*
- Design principles, in order to make work in the "art department" easier and more intuitive:
  1. Try to be physically correct, but intuitiveness of parameters has priority over correctness
  2. As few parameters as possible
  3. All parameter should be in [0,1]
  4. All parameters settings and combinations should be possible and lead to plausible results (no "funny" effects for specific values/combinations)
- Has still a lot more parameters than we have covered
  - But most can be modeled easily by introducing similar, additional terms

The Parameters



baseColor	baseColor = the surface color, usually supplied by texture maps
subsurface	subsurface scattering approximation
metallic	metallic-ness (linear blend between two different models
specular	incident specular amount
specularTint	a concession for artistic control that tints incident specular towards the base color
roughness	surface roughness, controls both diffuse and specular response
anisotropic	degree of anisotropy, controls the aspect ratio of the specular highlight
sheen	additional grazing component
sheenTint	amount to tint sheen towards base color
clearcoat	second, special-purpose specular lobe (für lackierte Oberflächen)
clearcoatGloss	0 = "satin" appearance, 1 = "gloss" appearance

Sources / Literature

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