JOURNAL OF COMBINATORIAL THEORY (B) 17, 234–239 (1974)

Maximum Matching and the Game of Slither

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Communicated by W. T. Tutte

Received June 27, 1973

Slither is a game played on a finite graph in which the players alternately choose edges so as to form a path. In this paper we present a strategy for Slither. The strategy depends upon an application of Edmond's maximum matching algorithm to the graph and a sequence of induced subgraphs. The strategy is practical in the sense that the amount of computation necessary has a polynomial bound.

1. INTRODUCTION

The game of Slither has been proposed by D. Silverman and discussed in two articles by M. Gardner [2]. In this paper we present a strategy for playing Slither; our results generalize a strategy due to R. Read [2] which applies in certain special cases.

Slither is a two-person game played on a finite undirected graph G with no loops or multiple edges. Player 1 chooses an edge of G; then each player in turn chooses a previously unchosen edge, subject to the restriction that the set of chosen edges must at all times form a path in G. The first player unable to move loses. This may result because there are no unchosen edges available at the ends of the path, or because each unchosen edge at the ends of the path meets the path at another point. Since the graph G is finite, the game will stop; no draws are possible.

Our solution to Slither is expressed in terms of maximum matchings in the graph G. An algorithm for finding maximum matchings has been given by Edmonds [1]; familiarity with his paper is a prerequisite for understanding our proofs.

In Section 2 we give some basic facts about matchings; these are given in more detail by Edmonds. The proof of Lemma 1 makes implicit use of some definitions and results from [1] not otherwise referred to in this paper.

In Section 3 we present our solution to Slither. The solution is given in Theorem 1; the strategy is given in Lemmas 2, 3, and 4 and in the proof of Theorem 1. In order to apply our strategy, it is necessary to be able to use Edmonds' algorithm.

2. Matchings in a Graph

A *matching* M in G is a set of edges such that no two edges of M meet at a vertex of G. For the pair (G, M), where M is a matching in G, a vertex v is called *covered* if v is a vertex of an edge $e \in M$; otherwise v is called exposed. A *maximum matching* is a matching which covers the maximum possible number of vertices.

An *outer vertex* of G is a vertex which is exposed in some maximum matching of G. An inner vertex of G is a vertex which is not an outer vertex of G but which is joined by an edge of G to an outer vertex. A *perfect vertex* of G is a vertex which is neither inner nor outer. One application of Edmonds' algorithm will find a maximum matching which leaves any desired outer vertex exposed; in addition, the algorithm will classify the vertices into inner, outer, and perfect vertices.

Let E denote the set of edges of the graph G. For $F \subset E$, let $\bar{F}$ denote the set of edges of E not contained in F. For $F_1, F_2 \subset E$, let $F_1 + F_2 = (F_1 \cap \bar{F}_2) \cup (F_2 \cap \bar{F}_1)$. An *alternating path* for the pair (G, M) is a path A such that the edges of A are alternately in M, in $\bar{M}$, etc. An alternating path A is called an *augmenting path* if the endpoints of A are exposed; clearly if A is an augmenting path, then $M + A$ is a matching which covers two more vertices than M, so that M is not maximum if an augmenting path exists. If A is an alternating path with one endpoint exposed and one endpoint covered, then $M + A$ is a matching which covers the same number of vertices as M; however, the exposed endpoint of A is now covered and vice versa.

Let W be a subset of the vertices of G; the *subgraph induced by W* contains the vertices of W and all edges of G joining them.

Lemma 1. *Let M be a maximum matching of G. Let A be an alternating path for (G, M) such that both end edges of A are contained in M. Then if one endpoint v of A is an inner vertex of G, vertex w, the other endpoint of A, must be an outer vertex of G.*

Proof. Since v is an inner vertex of G, v is an inner vertex of a Hungarian tree J for (G, M). The edge $e_1 \in M$ which meets v is an edge of J. Vertex v_1, the other vertex of e_1, is an outer vertex of J and hence an outer vertex of G. This proves the special case where A is the single edge e_1.

There is an alternating path B for (G, M) such that v_1 is one endpoint of B, with e_1 the corresponding end edge, and vertex r, the other endpoint of B, is exposed. To construct B, let M' be a maximum matching of G which leaves v_1 exposed—such an M' must exist since v_1 is an outer vertex of G. Then the component of $M + M'$ which includes v_1 is the desired alternating path.

Proceeding along B from r to v_1, let s be the first vertex of A encountered. Since both A and B are alternating paths, there is an edge $f \in M$ which meets s, moreover $f \in A \cap B$. Let B' be the subpath of B from r to s; let A' be the subpath of A from s to v, and A'' the subpath of A from s to w. If $f \in A'$, then $B' \cup A'$ is a planted tree for (G, M), and v is an outer vertex of $B' \cup A'$. Since M is maximum, v would then be an outer vertex of G, contradicting the hypothesis that v is an inner vertex of G. Therefore $f \in A''$; $B' \cup A''$ is a planted tree for (G, M), and w is an outer vertex of G. Q.E.D.

3. The Strategy for Slither

The basic idea in our strategy is for one player, say player i, to construct a maximum matching M, and with respect to M, force the path P of chosen edges to eventually have the following properties:

(1) Prior to each play of player j, either each end edge of P will be an edge in M, or one end edge of P will be an edge of M, and the opposite endpoint of P will be an exposed vertex.

(2) Prior to each play of player i, the vertex w, which is the endpoint that player j has just added to the path P, will meet an edge $m \in M$, and $m \cap P = w$.

If (1) holds prior to some play of player j, and if for any edge chosen by player j (2) will then hold, a winning strategy exists for player i. He merely chooses the edge m at each turn, thus returning the situation to (1); eventually player j will be unable to move. It is important to observe that during this play, player j will never be able to choose an edge in M, while player i will always choose an edge in M.

We will discuss three special cases in Lemmas 2, 3, and 4; the general strategy for Slither is given in Theorem 1.

Lemma 2. *Let M be a maximum matching for G and $e_0 \in M$ an edge joining two perfect vertices of G. Then if player* 1 *chooses e_0 at his first turn, he can force a win.*

Proof. After player 1 has chosen e_0, (1) holds at the first turn of player 2. To see that (2) will hold after player 2 has chosen an edge, consider the first time that (2) fails. Let f be the edge just chosen by player 2, and let P' be the subpath of P with end edges e_0 and f. If no edge $m \in M$ is available for player 1, then vertex w is exposed for M, and thus P' is an alternating path with one vertex exposed. It follows that $M + P'$ is a maximum matching which leaves a vertex of e_0 exposed, contradicting the hypothesis that e_0 joins two perfect vertices of G. Therefore an edge m is always available for player 1.

To see that $m \cap P = w$, observe that no vertices of P are exposed for (G, M). By the definition of matching, m cannot meet P except at w. Q.E.D.

LEMMA 3. *If player 1 initially chooses an edge e_0 such that one vertex v_0 of e_0 is an outer vertex of G, then player 2 can force a win.*

Proof. There is some maximum matching M which leaves v_0 exposed. Then vertex v_1, the other vertex of e_0, must be covered by an edge $f \in M$. Player 2 chooses f. Situation (1) now holds with $j = 1$. To see that (2) will always hold after player 1 has moved, consider the first time that (2) fails.

If no edge $m \in M$ is available for player 2, then one end of P is an exposed vertex $w \neq v_0$, and the subpath P' joining w to v_0 is an augmenting path for M; since M is maximum, no augmenting path can exist.

The edge m cannot meet P except at w, for vertex v_0 of P is exposed, and all other vertices of P are covered by edges in P. Q.E.D.

LEMMA 4. *Let the path R of edges chosen up to a particular point in the play contain only inner vertices of G, and let the edge e_0 join an endpoint of R to an outer vertex of G. Then if player j chooses edge e_0, player i can force a win.*

Proof. Let the endpoints of e_0 be v_0 and v_1, with v_0 an outer vertex of G and v_1 an endpoint of R; let v_2 be the other endpoint of R. (We specifically assume that $v_2 \neq v_1$, since if R had no edges, Lemma 3 would apply.) Let M be a maximum matching of G which leaves v_0 exposed; by Lemma 1, no edge of R is contained in M. Since vertex v_2 is an inner vertex of G, v_2 must meet some edge $f \in M$; let v_3 be the other vertex of f. Then $v_3 \neq v_0$, since v_0 is exposed, and $v_3 \notin R$ by Lemma 1. Player i chooses edge f, and now situation (1) holds for player j. As before, we consider the first time that (2) fails to hold for player i.

Let P' be the subpath of P starting at v_0 and going to the endpoint of P away from R. Let P'' be the subpath of P starting at v_2 and going through edge f to an endpoint of P. Then both P' and P'' are alternating paths—although P' might contain no edges— and w is an endpoint of either P' or P''.

If $w \in P'$ and no edge m is available for player i, then P' has both endpoints exposed and thus is an augmenting path for M; since M is maximum, no such path can exist, and thus an edge $m \in M$ is available.

Let z be the other vertex of m. By the construction of P, if z is a vertex of P, then z must be a vertex of R, since v_0 is exposed and all vertices added to P after v_0 are covered by edges in $M \cap P$. But if $z \in R$, then $P' \cup m$ is an alternating path with one endpoint exposed, and $M + (P' \cup m)$ is a maximum matching which leaves z exposed. Since all vertices of R are inner vertices of G, this situation cannot happen.

If $w \in P''$ and no edge m is available, then P'' is an alternating path with one endpoint w exposed and the other endpoint v_2 covered. Then $M + P''$ is a maximum matching which exposes v_2, contradicting the hypothesis that v_2 was inner vertex of G.

Let z be the other vertex of m. As before, if z is a vertex of P, then z must be a vertex of R. But if z were a vertex of R, then z would be an inner vertex of G and the path $P'' \cup m$ would contradict the conclusion of Lemma 1. Q.E.D.

THEOREM 1. *Let G be a finite graph. Let G_1 be the subgraph of G induced by the inner vertices of G. Inductively, let G_{i+1} be the subgraph of G_i induced by the inner vertices of G_i. The sequence G_i terminates. If some G_i contains a perfect vertex, then player 1 can force a win in the game of Slither; if no G_i contains a perfect vertex, then player 2 can force a win.*

Proof. By the definition of inner vertex, not all vertices of any G_i can be inner; thus G_{i+1} contains fewer vertices than G_i, and so the sequence terminates.

Now suppose that some first G_i contains a perfect vertex v. Let M_i be a maximum matching of G_i. Since v is a perfect vertex of G, some edge $e \in M_i$ meets v. By Lemma 1, w, the other vertex of e, must also be a perfect vertex of G_i. Player 1 should choose edge e and follow the strategy of Lemma 2, as applied to the graph G_i. Since player 1 would win the game played on G_i, player 2 will at some time either have no moves available—even allowing edges in the entire graph G— or play an edge $f \in G$ which is not in G_i. If player 2 has no moves available, then he loses. If he chooses an edge f which is not in G_i, let u be the other vertex of f. Let G_k be the graph of highest index in which vertex u appears.

Since vertex u is a vertex G_k but not a vertex of G_{k+1}, u must be either a perfect vertex of G_k or an outer vertex of G_k. By the choice of i, G_k has no perfect vertices; therefore u is an outer vertex of G_k. All other vertices of the path P are inner vertices of G_k, so that the hypotheses of Lemma 4 are met. Player 1 now follows the strategy of Lemma 4 on the graph G_k until player 2 either loses the game or chooses an edge not in G_k.

If player 2 chooses an edge not in G_k, Lemma 4 will again apply, this time to some graph G_j, $j < k$. The play continues in this fashion until finally, in the play on graph $G_0 = G$, player 2 has no escape.

On the other hand, suppose that no graph G_i contains a perfect vertex. Let edge e be the first edge chosen by player 1; then, in G_i, the highest index graph in which e appears, at least one vertex of e must be outer. Then by Lemma 3 player 2 has a winning position on the graph G_i. When player 1 plays an edge not in G_i, Lemma 4 applies as above, and thus player 2 wins. Q.E.D.

The strategy described in this theorem is practical in the sense that the amount of computation has a polynomial bound. Edmonds has shown that the computation in his algorithm has a polynomial bound; moreover if G has v vertices, then the number of inner vertices is less than $v/2$, and thus the number of applications of Edmonds' algorithm is $0(\log V)$.

A number of special cases of Slither are discussed in Gardner's second article; the application of Theorem 1 is immediate.

If G is the complete graph on n vertices, then for even n a perfect matching exists, and thus player 1 wins. For odd n, all vertices are outer, and player 2 wins.

If G is an $m \times n$ rectangular grid, then if mn is even, a perfect matching exists, and player 1 wins. If mn is odd, then all vertices at an even distance from a corner vertex are outer, and player 2 wins.

The author thanks Roger Shepherd for helpful comments during the preparation of this paper.

References

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