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Advanced Algorithms UE 2

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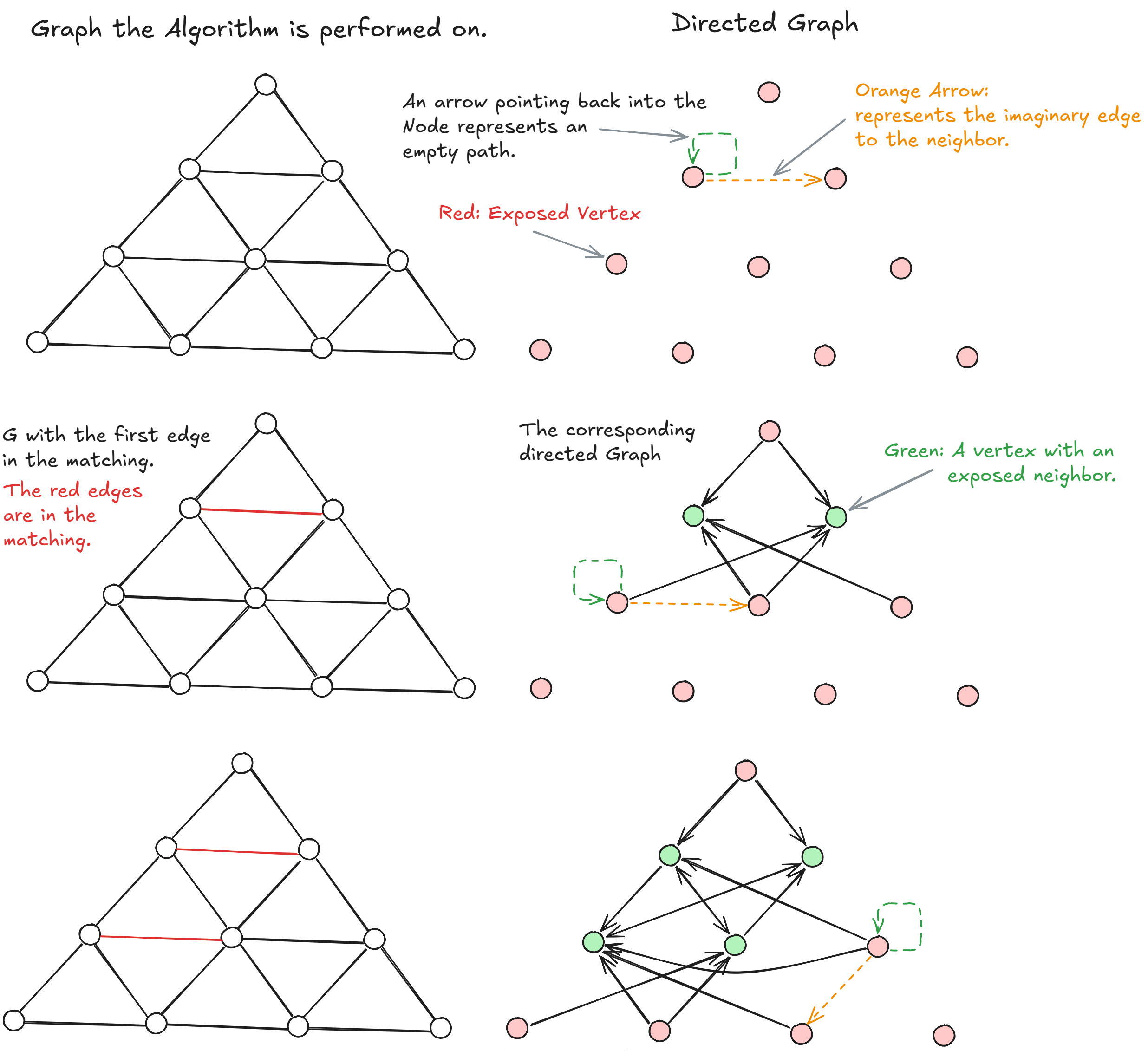
Group 5

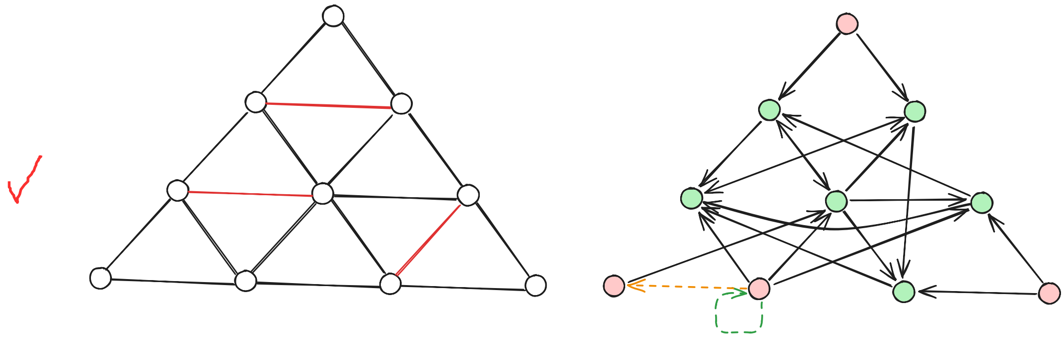
Exercise 2.1

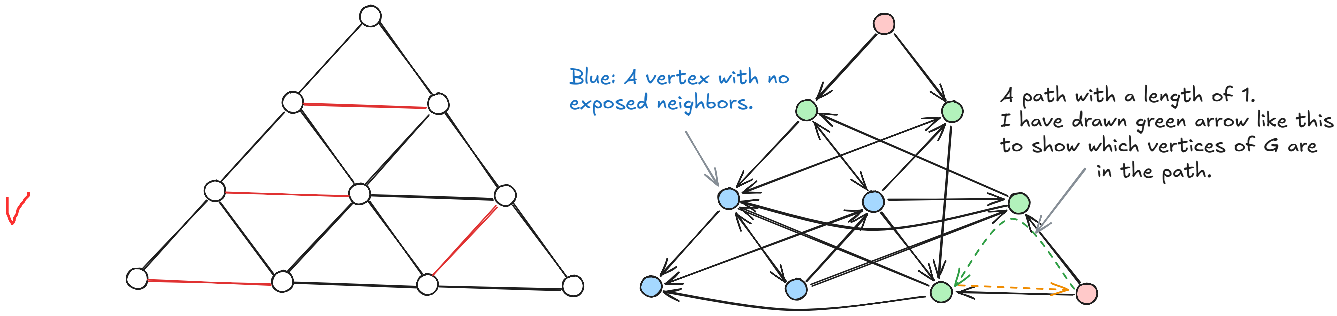
10/10

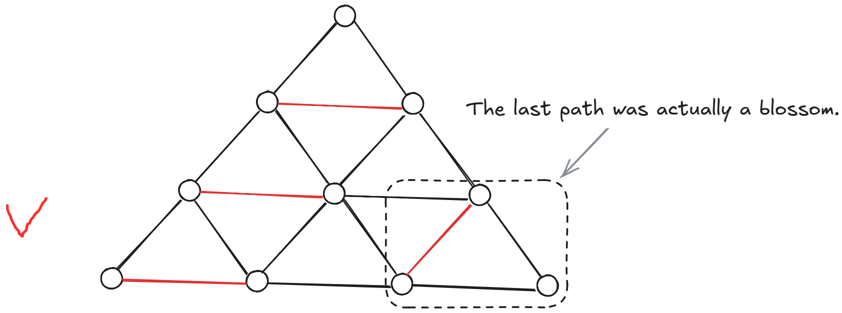
> Compute a maximum matching of the graph given below. Choose the shortest paths in DM such that you apply at least 3 times the 'shrink' operation of the algorithm.

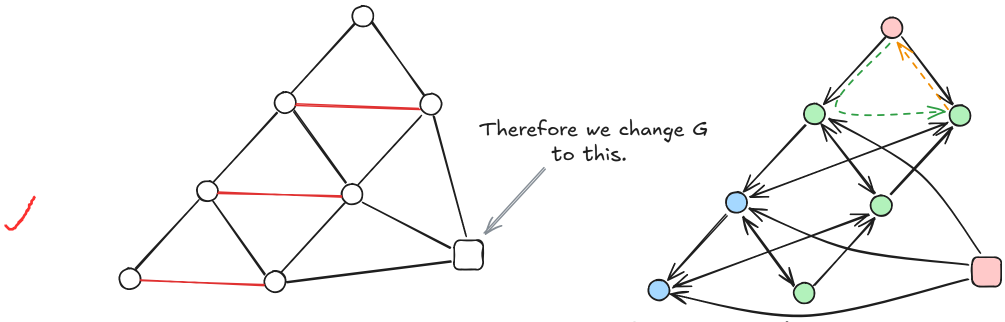


AA UE 2.1.excalidraw.svg









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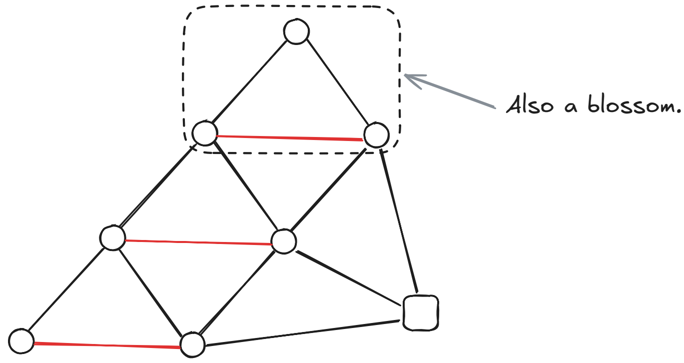
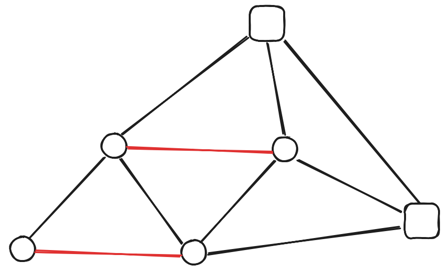
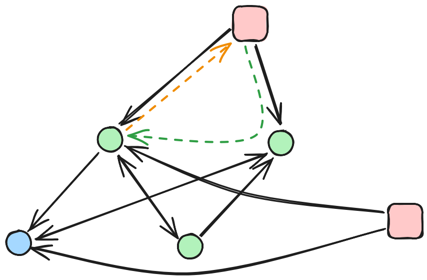
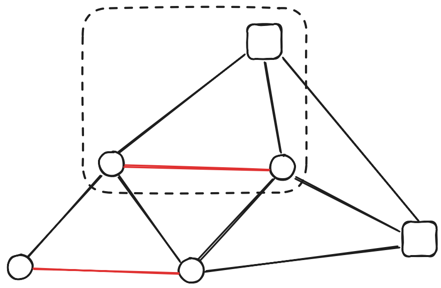
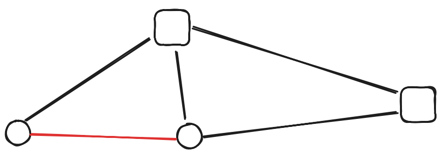


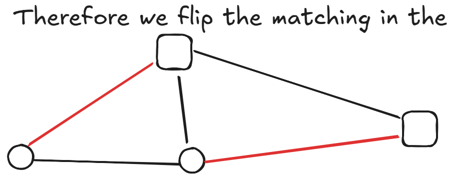




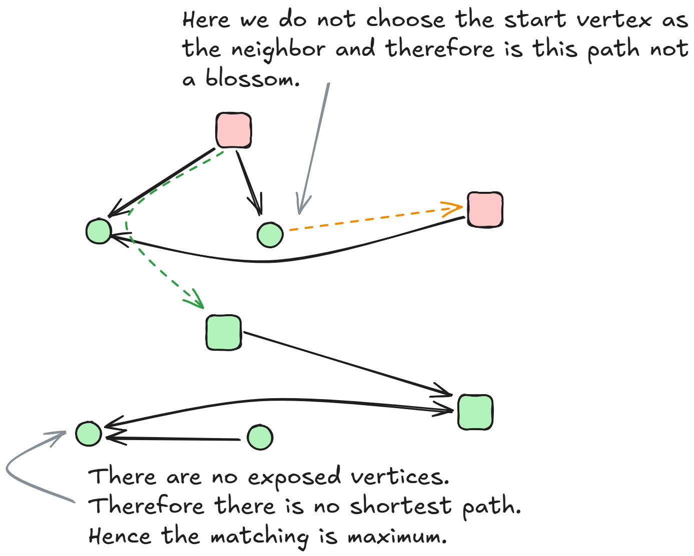




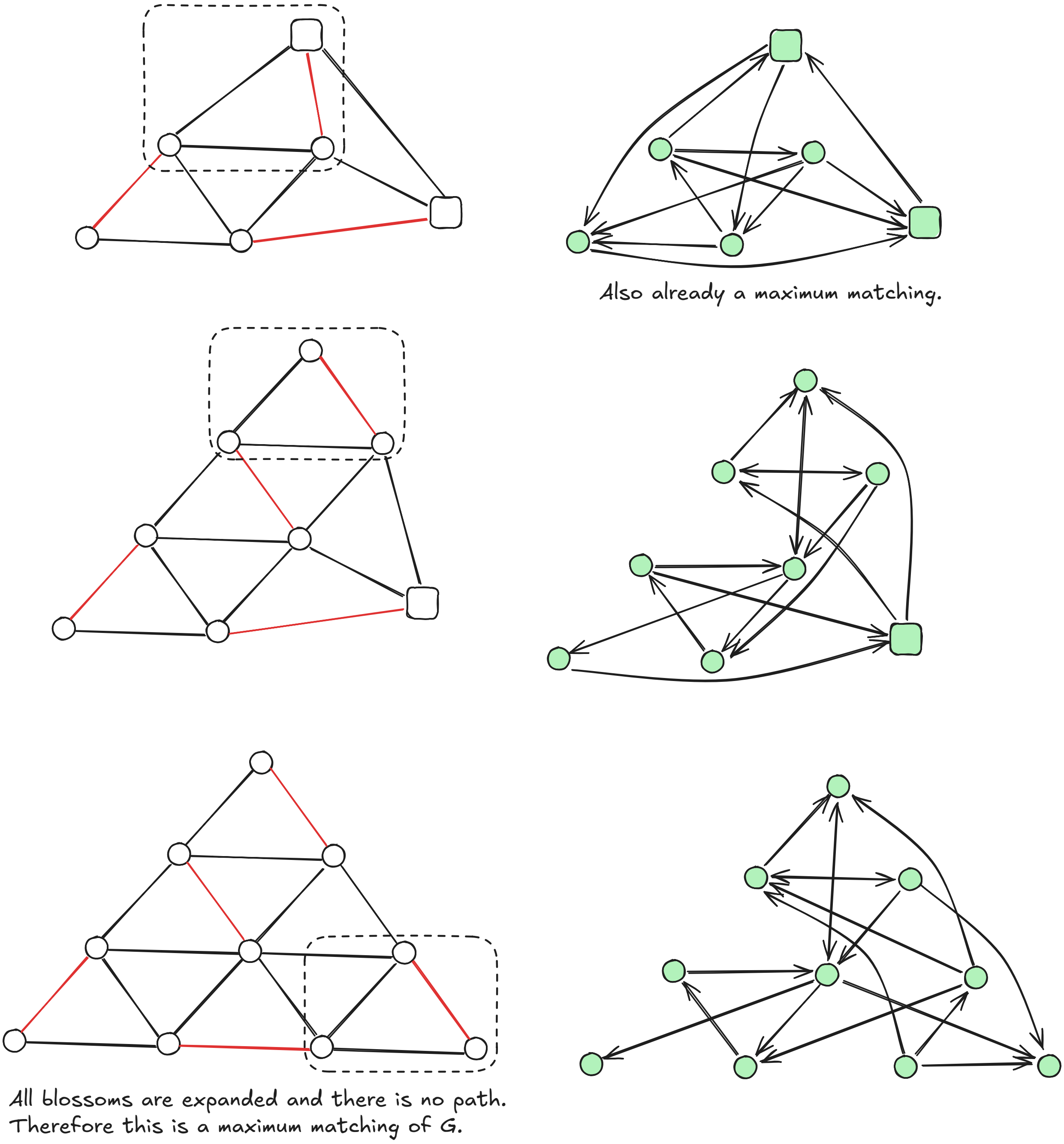
Therefore we flip the matching in the path.



We expand the blossoms recursively.



AA UE 2.1.2.excalidraw.svg



AA UE 2.1.3.excalidraw.svg

Exercise 2.2

> Consider the following simple algorithm to compute a matching in a given graph G.
> We fix some $k \in N, k \geq 1$.
>
> 1. Let M be the current matching.
> 2. If there is a set of k edges $M' \subseteq M$ and a set of $k+1$ edges $F \subseteq E \setminus M$ such that $(M \setminus M') \cup F$ is a matching, update M to

> $(M \backslash M') \cup F$.
> Show the following:

a) The running time of the algorithm is $O(|E^{2(k+1)})$.

Given:

Let $G = (V, E)$
Let M be a matching in G
Let $k \geq 1$

The Algorithm written in steps:

Go over all possible $M' \subset M \ : \ |M'| = k$
There are $\binom{|M|}{k}$ different M'.
Go over all possible $F \subseteq E \backslash M \ : \ |F| = k + 1$ ✓
There are $\binom{|E \backslash M|}{k+1}$ different F.
1/2 Check if $(M \backslash M') \cup F$ is a matching
This takes $k + 1$ operations
because we need to check if every new edge has no neigbor in the matching.

$O\left(\binom{|M|}{k} \cdot \binom{|E \backslash M|}{k+1} \cdot (k+1)\right)$ ✓
$\Rightarrow O\left(|E|^k \cdot |E|^{k+1} \cdot (k+1)\right)$ because $\binom{|M|}{k} \leq |E|^k$, $\binom{|E \backslash M|}{k+1} \leq |E|^{k+1}$
$\Rightarrow O\left(|E|^{2(k+1)} \cdot (k+1)\right)$ because $|E|^{k+1} \geq |E|^k$ why?
$\Rightarrow O\left(|E|^{2(k+1)}\right)$ because $k + 1 \leq |E|^{2(k+1)}$ O(k+1) = O(1) as k is constant
you need to repeat the procedure until no F can be found anymore, this gives another O(|E|)

b) Let M be the matching output by the algorithm and let M^* be a maximum matching in G. -1

> Show that $|M| \geq \frac{k+1}{k+2}|M^*|$

Lets take $M \triangle M^*$
Each connected component is either:

- a cycle: the number of edges from M and M^* are equal. √
- 1/4 a path: the number differs by 1 if it starts and ends with an edge from M^*, it has one more edge from M^* than from M.

Since no local improvement of exchanging k edges from M with $k+1$ edges from $E \setminus M$ is possible,
the number of M^* edges cannot exceed the number of M edges by more than a factor of $\frac{k+1}{k}$. why?

$(k+1)(|M^*| - |M|) \leq k|M|$
$\Rightarrow (k+1)|M^*| \leq (k+2)|M|$
$\Rightarrow |M| \geq \frac{k+1}{k+2}|M^*|$

Exercise 2.3

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> Consider the game Slither by W. Anderson.
> Given an undirected, simple connected graph $G = (V, E)$, two players choose in turns an edge e with the following rule:
> e was not yet chosen and the set of up to now chosen edges (including e) represents a simple path.
> The player who cannot choose an edge according to the rules loses.
> Show: If G contains a perfect matching, then there is winning-strategy for the first player.

Given

Let $G = (V, E)$
Let M be a perfect matching in G

Observation

If $M \subset P$ and both end edges are in M what is P?
-> no futher edge can be chosen.

Proof

An futher edge can only be choosen if it leads to a vertex not in P.
If the vertex would be in P, P would not be a simple path.
M covers all vertecies in G, because M is perfekt, perfect
therefore there is no edge that leads to a vertex not in P.

Winning strategy

The first player always chooses an edge in M.
Therefore the second player must allways always choose an edge not in M.
The second player will loose after the first player chooses the last edge in M
because $M \subset P$ and both end edges are in M.

Assumption

1. The first player allways choose an edge in M.
2. Both end edges are in M when the second player chooses.
3. P is an M alternating path.

Proof

by induction over $n = |P|$

Start $n = 0$

The first player chooses an edge in M. (1. Assumption)
-> 2. Assumption: both end edges are in M
-> 3. Assumption: the path contains only one edge in M

Step n' = n + 1

- If $n \nmid 2$ (the second player chooses)
  The second player must choose an edge $(x, v) \notin M$ (2. Assumption)
  therefore $v \in M$ (M is perfekt) and $v \notin P$ (P is simple).

-> 3. Assumption: The second player extended the end edges whitch are in M with an edge that is not in M.

- If $n \mid 2$ (the first player chooses)
  $\exists \ (v, u) \in M$
  $u \notin P$ because P would need to contain two cosequtive edges that are not in M. (3. Assumption)

-> 1. Assumption the first player can choose (v, u)
-> 2. and 3. Assumption: The player first will extend the end edge that is not M.

At every step in the induction all three assumptions are true.
I have also marked this with the arrows ->

Source

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