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Advanced Algorithms UE 1

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Exercise 1.1 (Algorithm for maximum weight bipartite matching)

> For the weighted bipartite graph given below and for each $k \in \{1, 2, 3, 4, 5\}$, compute a maximum weight matching M_k of cardinality k.

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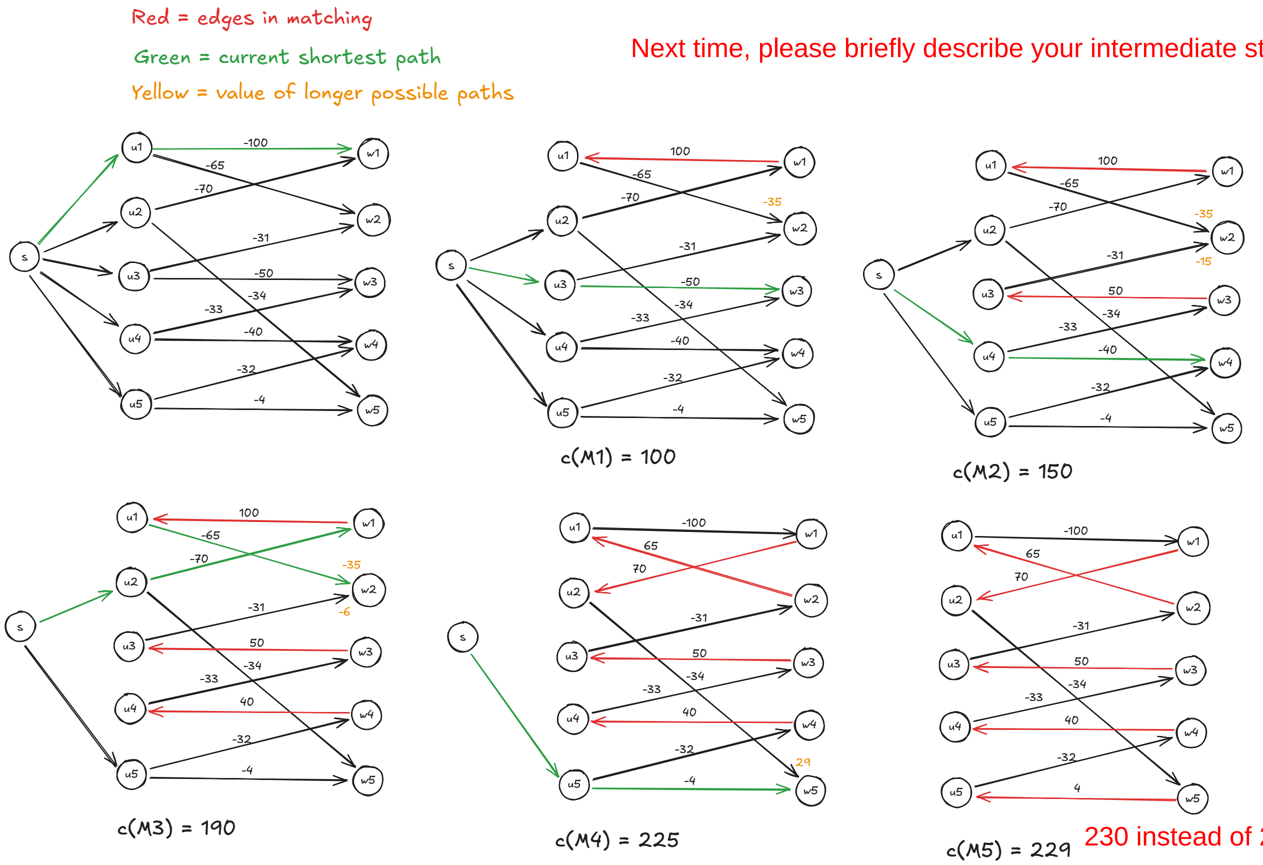
Next time, please briefly describe your intermediate steps



230 instead of 229

There is a shortest path (u5,w4,u4,w3,u3,w2,u1,w1,u2,w5) with cost -5 instead of -4.

-3

Exercise 1.2 (Improved running time for maximum weight bipartite matching)

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> Prove the following theorem from the lecture:
> We can compute a maximum-weight matching in a bipartite graph

> in time $O(n' \cdot (|E| + |V| log(|V|)))$, where n' is the minimum size of a maximum-weight matching.

Given

- bipartite graph $G = (V, E, c)$
- directed Graph D_M
- set of exposed vertecies $R_M \subsetneq R$
- the maximum weight matching M with no $s - R_M$ path in D_M with negative length

Task

Let $k := |M|$ be the cardinality of M
Let M_n be a maximum matching in G of cardinality $n \in k..|\frac{V}{2}|$ Wrong notation.

proof that $c(M_n) \leq c(M)$

Proof by induction over n

Start $n = k$

$M = M_n$ -> $c(M) = c(M_n)$
(M and M_k are equal so this is obviously true)

Step $n' = n + 1$

Let P be a $s - R_{M_n}$ path in D_{M_n}
Let P' be the shortest $s - R_{M_n}$ path in D_{M_n}
Let $c(P) := \sum_{e \in P} c(e)$ *(The sum of all weights of a path also refered to as the length.)*

Observation

$c(M_{n'}) = c(M_n) - c(P')$ *(The weight of the matching always increases by the negative weight of the shortest path. I have not proofen this futher as it seems obvious. I hope this is fine. I did not really know how formal the proof should be.)* Yes, be a bit more detailed next time. -1

proven

further

$c(M_{n'}) \leq c(M_n)$ because $c(P) \geq 0$

How do you now get the desired runtime? -1

Exercise 1.3 (Algorithm for the assignment problem)

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> Prove the following theorem from the lecture:
> We can compute a minimum-weight perfect matching in a bipartite graph in time $O(|V| \cdot (|E| + |V|log(|V|))$.

Given

- bipartite graph $G = (V, E, c)$

Task

1. Show how to perform the minimum-weight perfect matching in a bipartite graph
2. Proof that it has a runtime of $O(|V| \cdot (|E| + |V|log(|V|)))$

1. Show by reduction to maximum weight matching

Let $C := max(c(e)) \; \forall e \in E$
Let $c'(e) := C - c(e)$
Let $G' := (V, E, c')$
Perform the first and second step of maximum weight matching on G'.
Let S be the set of computed matchings from the second step.
Let $k := \frac{|V|}{2}$
Let M be the matching in S with cardinality k. ✓
M is the minimum-weight perfect matching in G.

Proof: M exists in S

A maximum weight matching of G could be of cardinality k therefore the S must contain M.

Proof: M is a perfekt matching

M is of cardinality k.

Proof: M is a minimum-weight matching

Observation

$$\sum_{e \in M} c'(e) = C \cdot |M| - \sum_{e \in M} c(e)$$

$\sum_{e \in M} c(e)$ is the maximum weigth of all perfect matchings.
If the maximum value is subtracted from a constant value the resulting value is the minimum. Because both sums are over the same matching,
$\sum_{e \in M} c'(e)$ is the minimum weight of all perfect matchings.
-> M is a minimum-weight matching

2. Proof that its runtime is $O(|V| \cdot (|E| + |V| log(|V|)))$

Steps	Runtime
Let $C := max(c(e)) \, \forall e \in E$	$O(E)$
Perform maximum weight matching	$O(\|V\| \cdot (\|E\| + \|V\| log(\|V\|)))$

All other steps have a runtime of $O(1)$.

$O(|V| \cdot (|E| + |V|log(|V|)) + E)$
$\Rightarrow O((|V| + 1) \cdot (|E| + |V|log(|V|))$
$\Rightarrow O((|V|) \cdot (|E| + |V|log(|V|))$

Sources

The idea for C I got from here: https://www.cse.iitd.ac.in/~naveen/courses/CSL851/lec4.pdf (10.4.25, 22:09)