Prof. Dr. Nicole Megow
Dr. Felix Hommelsheim
Dr. Alexander Lindermayr
Bart Zondervan

Summer 2025

Advanced Algorithms

Exercise Sheet 4

Submission: Monday, May 12, 2025, at 11:59 am.
This exercise will be discussed on Wednesday, May 14, 2025.

Exercise 4.1 (Exponential running time of Ford-Fulkerson) (6 Points)

a) Give an example of a network $\mathcal{N}$ in which $|V| \leq 4$ and the number of iterations of the Ford-Fulkerson Algorithm is c_{max}, where c_{max} is the largest capacity of any arc.

b) What is the size of the input for this example?

c) Prove that this leads to an exponential running time (w.r.t. the size of the input).

Exercise 4.2 (Adding flows) (8 Points)

Let f be a flow in a network $\mathcal{N} = (V, E, c, s, t)$, and let g be a flow in the residual network $\mathcal{N}_f$ of $\mathcal{N}$ w.r.t. f. Prove the following statements:

a) $f + g$ is a flow in $\mathcal{N}$.

b) If g is a maximum flow in $\mathcal{N}_f$, then $f + g$ is a maximum flow in $\mathcal{N}$.

Exercise 4.3 (Decomposition of flows) (6 Points)

Let $\mathcal{N} = (V, E, c, s, t)$ be an s-t-network with m edges. Prove that there exists a maximum flow in $\mathcal{N}$ that is the sum of at most m flows $f_1, \ldots, f_k$, each of which takes positive values only on a single s-t-path in $\mathcal{N}$. Moreover, prove that if all capacities in $\mathcal{N}$ are integers, then $f_1, \ldots, f_k$ can be chosen as integral flows.

The following exercises are no homework and shall be discussed in presence.

Exercise 4.4 (Multiple sources and sinks)

Let $G = (V, A)$ be a digraph with capacities $c\colon A \to \mathbb{N}$. Consider two disjoint sets S and T of vertices in G. An (S, T)-flow in (G, c) is a function $f\colon A \to \mathbb{N}$ such that

1. $0 \leq f(e) \leq c(e)$ for all $e \in A$, and
2. $\sum_{e\in\delta^+(v)} f(e) = \sum_{e\in\delta^-(v)} f(e)$ for all $v \in V \setminus (S \cup T)$.

The value of the (S, T)-flow f equals $\sum_{t\in T} \Big(\sum_{e\in\delta^-(t)} f(e) - \sum_{e\in\delta^+(t)} f(e)\Big)$.
Prove that an (S, T)-flow in (G, c) of maximum value can be computed in $O(m^2 \cdot n)$, where $m = |E|$ and $n = |V|$.

Exercise 4.5 (Intersection/union of minimum cuts)

Let $\mathcal{N} = (V, A, c, s, t)$ be a network, and let $\delta^+(X)$ and $\delta^+(Y)$ be minimum s-t-cuts in (G, c). Show that $\delta^+(X \cap Y)$ and $\delta^+(X \cup Y)$ are also minimum s-t-cuts in (G, c).

Exercise 4.6 (Minimum edge cover formula)

Let $G = (V, E)$ be a graph. An *edge cover* of G is a set of edges $F \subseteq E$ such that for every vertex $v \in V$ there exists an edge in F incident to v. Denote by $\rho(G)$ the size of a smallest edge cover in G, and by $\nu(G)$ the size of a maximum matching in G.
Prove that for any graph G without isolated vertices that $|V| = \nu(G) + \rho(G)$.