University of Bremen

FB3: Mathematik/Informatik

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Advanced Algorithms

Exercise Sheet 6

Submission: Wednesday, May 28 at 11:59 am.
This exercise will be discussed on Wednesday, June 4.

Exercise 6.1 (Flows in strongly connected networks) (10 Points)

Let $\mathcal{N} = (V, E, c, s, t)$ be an s-t network with the following properties:

- The subgraph of (V, E) induced by $V \setminus \{s, t\}$ is strongly connected. That is, for every $u, v \in V \setminus \{s, t\}$ there is a directed path from u to v in $(V \setminus \{s, t\}, E)$.
- We have that $\sum_{e \in \delta^+(s)} c(e) = \sum_{e \in \delta^-(t)} c(e) =: B$.
- For all $e \in E \setminus (\delta^+(s) \cup \delta^-(t))$, we have $c(e) = B$.

Prove that there exists a maximum flow f in $\mathcal{N}$ with $val(f) = B$. Moreover, give an algorithm that computes such a flow in time $O(|E|)$.

Exercise 6.2 (The FIFO Preflow-Push Algorithm) (10 Points)

A FIFO (first-in first-out) queue Q is a data structure where elements are added (push) to the end of the queue and removed from the front (pop). Consider the following variant of the generic Preflow-Push algorithm:

Algorithm 1: The FIFO Algorithm

```
Input: An s-t-network N = (V, E, c, s, t)
Init(N)
FIFO queue Q ← neighbors of s (all are active now)
while Q is not empty do
    u ← pop(Q)
    Discharge(u)
    if u is active then
        push(Q, u)
return f
```

Prove that the FIFO Algorithm computes a maximum flow in time $O(n^3)$, where $n = |V|$. To do so, you can use the following hints:

- Split the execution of the algorithm into phases, where the Phase 1 is the processing of the neighbors of s, Phase 2 is the processing of those vertices that are added to Q during Phase 1, Phase 3 is the processing of those vertices that are added to Q during Phase 2, and so on.

- Your goal is to show that the number of non-saturating `Push` operation is in $O(n^3)$ (why?). Such a bound follows from the number of phases (why?).

- To analyze the number of phases, use the potential function $\Phi = \max\{d(v) \mid v \text{ is active}\}$. Moreover, distinguish between phases without any relabeling and phases with at least one relabeling.