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Advanced Algorithms

Exercise Sheet 9

Submission: Monday, June 23 at 11:59 am.
This exercise will be discussed Wednesday, June 25

Exercise 9.1 (MST Modeling and Separation) (8 Points)

Consider the Minimum Spanning Tree (MST) problem over a graph $G = (V, E)$ with edge costs c_e and variables $x_e \in \{0, 1\}$ indicating whether edge e is in the solution. Use the so-called subtour elimination constraint

$$\sum_{e \in E(S)} x_e \leq |S| - 1 \quad \forall S \subset V,\ 2 \leq |S| \leq |V| - 1$$

and one global constraint:

$$\sum_{e \in E} x_e = |V| - 1.$$

(a) Give an ILP formulation for the MST problem and argue why it is correct.

(b) Give a polynomial-time separation algorithm for the LP-relaxation of your ILP.
Hint: Use your knowledge about flow or cut problems.

Exercise 9.2 (Total Unimodularity) (4 Points)

Let $G = (V, E)$ be a bipartite graph. Prove that the node-edge incidence matrix of G is totally unimodular.
Hint: Use one of the properties stated in class.

Exercise 9.3 (Complementary Slackness) (8 Points)

Dualize the following LPs and test with complementary slackness whether the given solutions are optimal.

(a) Solutions: $x = (0, 0, -9)^T$ and $y = (\frac{4}{9}, \frac{19}{9}, \frac{1}{9})^T$

$$\begin{array}{lrcrcrcr}
\max & x_1 & + & 3x_2 & + & x_3 & & \\
\text{s.t.} & x_1 & + & 2x_2 & + & 7x_3 & \leq & -3 \\
 & & & x_2 & - & x_3 & = & 9 \\
 & 9x_1 & & & & & \leq & 5 \\
 & x_1 & & & & & \geq & 0 \\
 & & & x_2 & & & \leq & 0
\end{array}$$

(b) Solutions: $x = (\frac{15}{7}, -\frac{11}{7})^T$ and $y = (\frac{4}{7}, \frac{23}{14}, 0)^T$

$$\begin{array}{lrcrcr} \min & 5x_1 & + & 6x_2 & & \\ \text{s.t.} & 3x_1 & - & x_2 & \geq & 8 \\ & 2x_1 & + & 4x_2 & = & -2 \\ & 3x_1 & + & 2x_2 & \leq & 4 \\ & x_1 & & & \geq & 0 \end{array}$$

Problems for solving in class on June 18th

Exercise 9.4 (Shadow Prices)

A furniture manufacturer produces three types of products: desks, tables, and chairs. Each product requires resources: wooden boards, sanding hours, and carpentry hours. The company has limited availability of each resource.

Available Resources

- 48 units of wooden boards
- 20 hours of sanding
- 8 hours of carpentry

Profit per Unit

- Desk: 60 Euro
- Table: 30 Euro
- Chair: 20 Euro

Resource Consumption per Product

Product	Wood	Sanding	Carpentry
Desk	8	4	2
Table	6	2	1.5
Chair	1	1.5	0.5

(a) Formulate the *primal LP* to maximize profit, using decision variables x_1, x_2, x_3 for the number of desks, tables, and chairs to produce, respectively.

(b) Formulate the *dual LP*, where a buyer offers to purchase your available resources. Interpret the dual variables y_1, y_2, y_3 as *shadow prices* (value per unit of resource).

(c) Using the optimal solutions:

$$x^* = (2, 0, 8) \quad \text{and} \quad y^* = (0, 10, 10)$$

answer the following:

- Which resource is *not fully used*, and what does this imply for its shadow price?
- Why does the optimal solution avoid producing tables?
- Why must the dual constraint corresponding to the desk be tight?

Exercise 9.5 (Total unimodularity)

(a) Show or disprove that the following matrices are totally unimodular.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

(b) What do you know about the set of optimal solutions of the linear program

$$\min_{x \in \mathbb{R}^4} \{c^T x \mid Bx \leq b\}$$

with $b, c \in \mathbb{Z}^4$?

(c) What do you know about the set of optimal solutions of the linear program

$$\min_{x \in \mathbb{R}^3} \{c^T x \mid Ax \leq b\}$$

with $b, c \in \mathbb{Z}^3$?

Exercise 9.6 (Solving an LP)

Given the primal LP:

$$\begin{array}{lrrrcl} \min & 5x_1 & +12x_2 & +2x_3 & & \\ \text{s.t.} & 2x_1 & +4x_2 & -8x_3 & \leq & 1 \\ & -8x_1 & +4x_2 & -x_3 & \geq & 0 \\ & x_1 & +2x_2 & +2x_3 & = & 5 \\ & x_1, & & x_3 & \geq & 0 \end{array}$$

(a) Construct the dual.

(b) Let $x^* = (0, 0.5, 2)^T$ be an optimum solution of the primal program. Construct an optimum solution of the dual program using complementary slackness.

(c) Verify strong duality with x^* and the optimum solution you found for the dual.

Exercise 9.7 (Modeling the Traveling Salesman Problem)

You are given a set of n cities, labeled $\{1, 2, \ldots, n\}$. For each pair of distinct cities i and j, there is a known cost c_{ij} to travel from city i to city j. The goal is to find a tour of minimum total cost that visits each city exactly once and returns to the starting city.

Model this problem as an integer linear program (ILP) using variables $x_{ij} \in \{0, 1\}$ indicating that the edge $\{i, j\}$ is part of the tour.

Exercise 9.8 (Modeling the Vertex Coloring Problem)

Let $G = (V, E)$ be an undirected graph. In a valid coloring, adjacent vertices must receive different colors. More precisely, a *valid coloring* of G with (at most) k colors is a function $f : V \to \{1, \ldots, k\}$ such that $f(v) \neq f(u)$ for all $\{v, u\} \in E$. A minimum vertex coloring in a given graph G is a valid coloring of G using the minimum number k^* of colors, meaning there must not exist a valid coloring with $k^* - 1$ colors. Model the problem of computing a minimum vertex coloring as an ILP.

Hint: If you use natural numbers as colors, then the largest color used in your coloring provides an upper bound on the number of colors used.