Prof. Dr. Nicole Megow
Dr. Felix Hommelsheim
Dr. Alexander Lindermayr
Bart Zondervan

Summer 2025

Advanced Algorithms

Exercise Sheet 10

Submission: Monday, June 30 at 11:59 am.
This exercise will be discussed Wednesday, July 2

Exercise 10.1 (5 Points)

Let $D = (V, A)$ be a directed graph and let $S \subseteq V$ be a subset of vertices. For each $s \in S$ let k_s be a natural number. Show that $(A, \{F \subseteq A : |\delta^-_F(s)| \leq k_s \; \forall s \in S\})$ is a matroid.

Exercise 10.2 (Intersection of matroids) (2+5 Points)

The *intersection* of two independent systems $(E, \mathcal{F}_1)$ and $(E, \mathcal{F}_2)$ is defined as $(E, \mathcal{F}_1 \cap \mathcal{F}_2)$.

(a) Show that $(E, \mathcal{F}_1 \cap \mathcal{F}_2)$ is again an independent system.

(b) Show that any independent system $(E, \mathcal{F})$ is the intersection of a finite number of matroids.

Exercise 10.3 (3+5 Points)

There are n tasks to be completed in n days. Each task j has a deadline $d_j \leq n$ by which it must be completed, and requires one day to process.

(a) We call a subset X of tasks *feasible* if there exists a schedule that can complete all tasks in X by their respective deadlines. Furthermore, we define

$$X(t) := \{j \in X \mid d_j \leq t\}$$

for all $t \in \mathbb{N}$. Show that $X \subseteq [n] = \{1, \dots, n\}$ is feasible if and only if for all $t \in \mathbb{N}$ it holds that $|X(t)| \leq t$.

(b) Show that $([n], \mathcal{I})$, where $\mathcal{I}$ contains all feasible subsets of $[n]$, is a matroid.

Exercises for solving in class on June 25th

Exercise 10.4

Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid with rank function r. Define

$$\mathcal{I}' = \{I \subseteq E : \exists \text{ basis } B \text{ of } \mathcal{M} \text{ such that } B \subseteq E \setminus I\}.$$

a) Show that $\mathcal{M}' = (E, \mathcal{I}')$ is a matroid. We call $\mathcal{M}'$ the *dual* matroid of $\mathcal{M}$.

b) Show that the dual matroid of $\mathcal{M}'$ is identical to $\mathcal{M}$.

c) Let r' be the rank function of $\mathcal{M}'$. Show that for all sets $S \subseteq E$,

$$r'(S) = |S| + r(E \setminus S) - r(E).$$