Discrete Mathematics

Contents

1 Enumerative combinatorics

1.1 Double counting

Lemma 1.1 (Handshaking lemma): *If N people shake hands, the number of those who shook an odd number of hands is even.*

Proof: Let $\mathcal{S}$ be the set of pairs of persons and handshakes (p, h). So $\mathcal{S} \subseteq P \times H$.

Now determine the cardinality of $\mathcal{S}$:

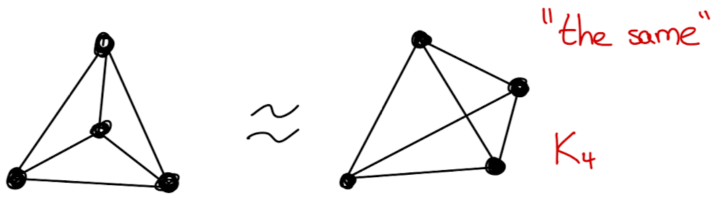
$|\mathcal{S}| = 2|H| = \sum_{p \in P} \text{Number of handshakes of } p$ Therefore the number of $p \in P$ with odd number of handshakes is even. ■

Notation 1.2: $\binom{S}{k}$ describes all subsets of a set S of cardinality k, for $k \in \mathbb{N}$.

Definition 1.3 (Simple graph): A *simple graph* G is a pair (V, E) where V is a set of vertices and $E \subseteq \binom{V}{2}$ set of edges. We call G *finite* if V is finite.

Definition 1.4 (Graph isomorphism): Two graphs G, H are *isomorphic* iff

$$\exists \varphi : V(G) \to V(H) \text{ such that } \varphi \text{ induces a bijection of } E(G) \text{ to } E(H)$$



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Convention 1.5: $[n] = \{1, ..., n\}$

Example 1.6 (Petersen's graph):

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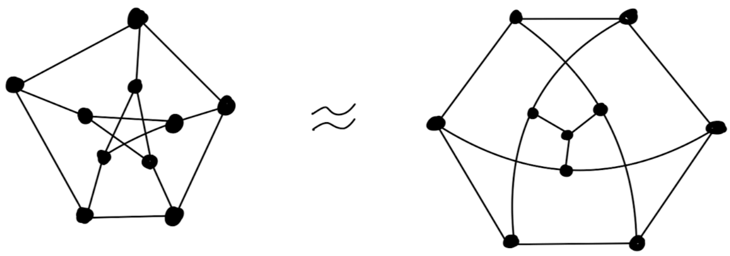
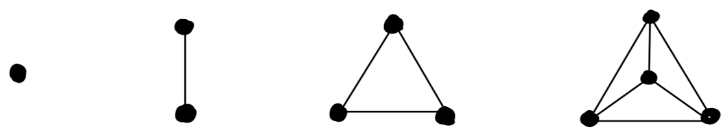


graph of all 2-subsets of $\{1, 2, 3, 4, 5\} = [5]$ edge if disjoint

From here on graphs are up to isomorphism.

Notation 1.7:

(1) <u>complete graph</u>: K_N with $1 \leq n$ vertices, all edges



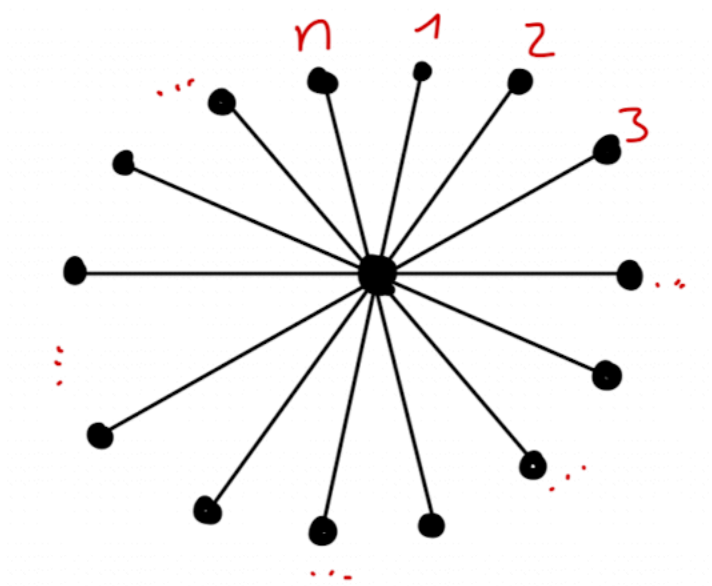
:

(2) <u>cycle graph</u>: C_n $n \geq 3$ vertices-classes $\operatorname{mod} n$
$0, 1, 2, ..., n-1$ $(x, x+1)$ are edges

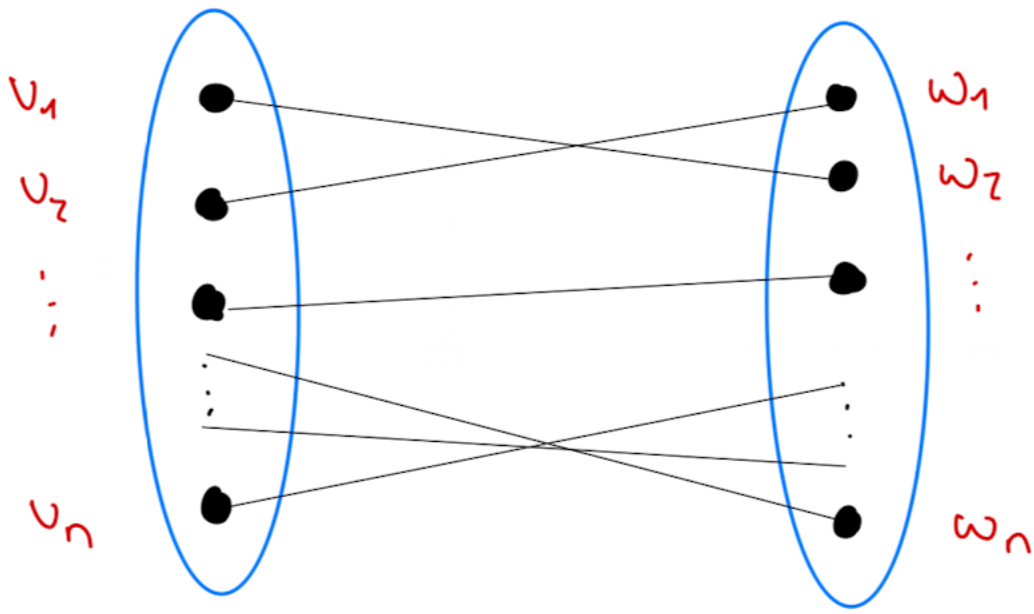
(3) <u>empty graph</u>: $V = \emptyset$

(4) <u>N_n-graph</u>: n vertices and no edges ($E = \emptyset$)

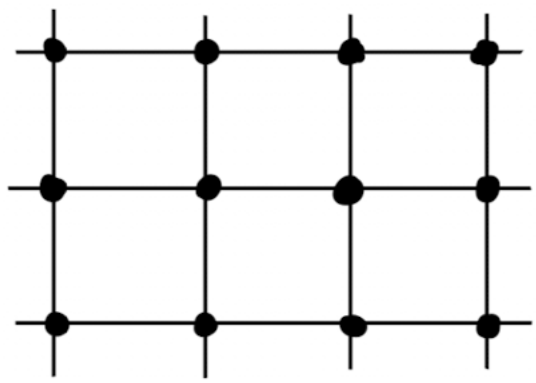
(5) <u>star graph</u>:



(6) $K_{m,n}$: for $m, n \geq 1$ edges = all (v_i, w_j) # edges = $m \cdot n$ complete bipartite graph



(7) Integer grid graph: vertices = (x, y) for $x, y \in \mathbb{Z}$ edges: $((x, y), (x+1, y))((x, y)(x, y+1))$



(8) Vector space W S-selected set of vectors Γ_s-gridgraph vertices= which are $\mathbb{Z}$-linear combination of vectors from S. edges: $(v, v + s), v \in V(\Gamma_s), s \in S$

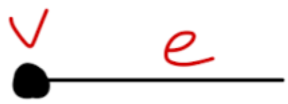
Remark 1.8 (Number of edges in K_n):

$$2|E| = \sum_{i=1}^{n} (n-1) = n(n-1)$$
$$\Leftrightarrow |E(K_n)| = \frac{n(n-1)}{2}$$

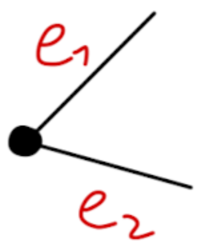
Definition 1.9 (Adjacency): Let G be a graph

(1) $v, w \in V(G)$ adjacent if $(v, w) \in E(G)$

(2) $v \in V(G)$ and $e \in E(G)$ are adjacent if $v \in e$



(3) $e_1, e_2 \in E(G)$ adjacent



Definition 1.10 (Valency): For a graph $G, v \in V(G)$

$$\text{val } v = \# \text{ adjacent edges to } v$$

Definition 1.11: A graph G is called *regular* if all vertices have the same valency d-regular.

Example 1.12:

(1) K_n is $(n-1)$-regular
(2) C_n is 2-regular
(3) $K_{n,n}$ is n-regular

edges of a d-regular graph with n vertices $= \frac{n \cdot d}{2}$

Corollary 1.13: *If d is odd, then n is even*

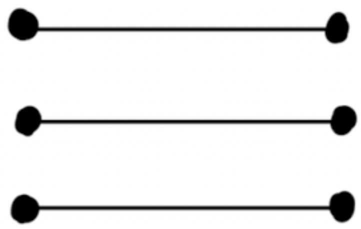
Example 1.14: $\nexists$3-regular graph with 2025 vertices

Definition 1.15: Let G be a graph. The *complement* of G is $\overline{\mathrm{G}}$: $V(\overline{\mathrm{G}}) = V(G)$
$E(\overline{\mathrm{G}}) = \{(v,w) | v, w \in V, (v,w) \notin E(G)\}$

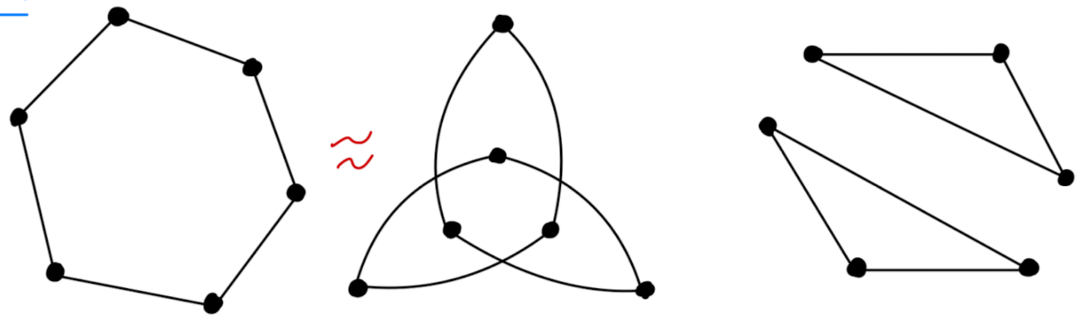
Self- complementary graph $G \approx \overline{\mathrm{G}}$

n=6: all d-regular graphs on 6 vertices

d = 1:

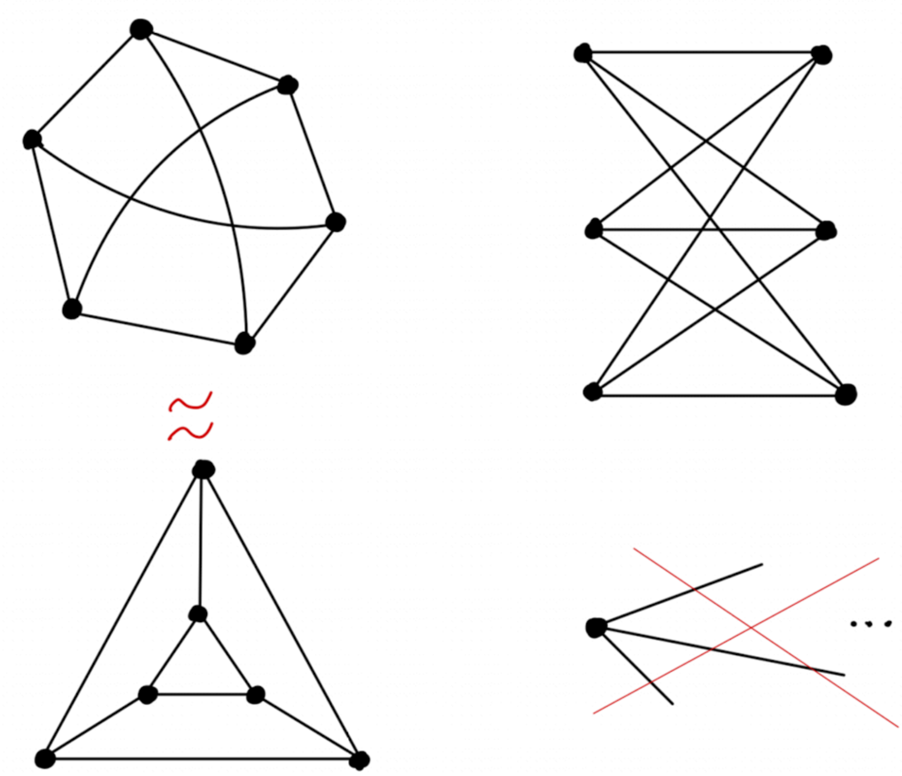


d = 2:



2 graphs

d = 3:



G-d-regular graph on n vertices $\overline{\mathrm{G}}$-$(n-d-1)$-regular graph $n = 6$: 3-regular graphs = complements of 2-regular graphs

Theorem 1.16: *Let G be a finite simple graph, then #vertices with odd valency is even.*

1.2 Binomial coefficients

Definition 1.17: For $n \geq k \geq 0$:
"n chose k" $\binom{n}{k}$-binomial coefficient = #k-subsets of $[n]$ $\binom{n}{0} = 1$, $\binom{n}{1} = n$, $\binom{n}{2} = \frac{n(n-1)}{2}$
$\binom{n}{k} = \binom{n}{n-k}$

Theorem 1.18: $\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$

Proof: Proof by obviousness. *(Ist halt die Potenzmenge lol)* ■

Proposition 1.19: *Let n be a positive integer #permutation of $[n] = n!$.*

Theorem 1.20: *Given $k!(n-k)!\binom{n}{k} = n!$ and $0! = 1$ we have (???)*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n-k+1}{k!} \cdot n$$

Proof: Double count permutations
#permutations $= n!$
$\pi_1 \pi_2 \ldots \pi_n$
choose the set $\{\pi_1, \ldots, \pi_k\}$

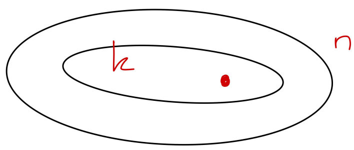
order them $k!$
order the rest $(n-k)!$
$\pi_1 \dots \pi_k \pi_{k+1} \dots \pi_n$ ■

Notation 1.21: $C(n,k) = \binom{n}{k}$ "Combinations"

Theorem 1.22:
(1) Pascal triangle rule

$$\binom{n+1}{k} = \underbrace{\binom{n}{k-1}}_{\text{subsets with } n+1} + \underbrace{\binom{n}{k}}_{\text{subsets without } n+1} \quad : 1\ 2 \dots n\ (n+1)$$

(2) $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$ $k\binom{n}{k} = n\binom{n-1}{k-1}$ *#k-subsets with a chosen leader*
(3) $\binom{k}{k} + \binom{k+1}{k} + \dots + \binom{n}{k} = \sum_{t=k}^{n} \binom{t}{k} = \binom{n+1}{k+1}$
(4) $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \dots + n\binom{n}{n} = n \cdot 2^{n-1}$ *??????????* $k\binom{n}{k}$
(5) $\binom{n}{0}^2 + \binom{n}{1}^2 + \dots + \binom{n}{k}^2 + \dots + \binom{n}{n}^2 = \binom{2n}{n}$
(6) *Triple counting*: $k\binom{n}{k} = n\binom{n-1}{k-1} = (n-k+1)\binom{n}{k-1}$



k-sets with a chosen leader Was bedeutet dieses :1 2 3 … 2n am Ende immer?? möglichkeiten aus den er auswählt

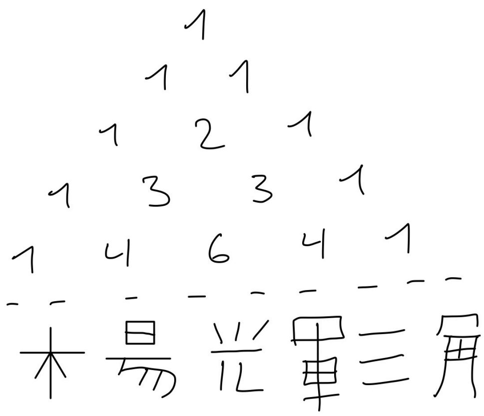
Proof:
(3) Choose $k+1$ elements out of $n+1$ elements by first choosing the maximal one $m = k+1, k+2, \dots, n+1$

■

Proposition 1.23: *For numbers* $n \geq k \geq l \geq 0$

$$\binom{n}{k}\binom{k}{l} = \binom{n}{l}\binom{n-l}{k-l} = \binom{n}{k-l}\binom{n-k+l}{l}$$

Proof: "Actually the same" 🤡 with $|K| = k, |L| = l$.
Counting $K, L, L \subseteq K$ Notice the prior example is a special case with $l = 1$. ■



Pascal Triangle was known in China before under the name 楊輝三角. According to Wikipedia also in ⛳ Persia and 🦩 Iran before that already.

Theorem 1.24 (Vandermonde's identity): *With $m, n \geq r$:*

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}$$

Proof:
- choose r people
- k men
- $r - k$ $\neg$men
- sum overall $k = 0, ..., r$

■

Theorem 1.25 (Binominal theorem):

$$\begin{aligned}(x+y)^n &= x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + ... + \binom{n}{n} y^n \\ &= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \text{ true symbolically}\end{aligned}$$

Proof: Coefficient of $x^k y^{n-k}$:

$$\underbrace{(x+y)(x+y) \cdot \dots \cdot (x+y)}_{n}$$

$\binom{n}{k}$ is the # ways to pick k brackets out of n. ■

Remark 1.26: For $x = y = 1$ it holds

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

Essentially just counting We can use Theorem 1.25 to generate new formulas. For $x = 1$ and $y = -1$ it holds $x + y = 1 - 1 = 0$, therefore

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - ... + (-1)^n \binom{n}{n}.$$

This can be shown by calculating the binomials for odd n and even n.

On the other hand for $x = 1$ and $y = 2$ it holds

$$3^n = \binom{n}{0} + 2\binom{n}{1} + 4\binom{n}{2} + 8\binom{n}{3} + \ldots + 2^n\binom{n}{n}.$$

Corollary 1.27: $(1+t)^n = \sum_{k=0}^{n} \binom{n}{k} t^k$

Corollary 1.28:

$$(x+y)^n(x+y) = (x+y)^{n+1}$$

Hier waren wir kurz raus:

$(x+y)^{2n} = ((x+y)^n)^2 = (x+y)^n(x+y)^n$ Compare the coefficient of $x^n y^n$

$$\begin{aligned} &\text{LHS: } \binom{2n}{n} \\ &\text{RHS: } \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} \cdot \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j \\ &\qquad (+ \cdots +) \qquad\qquad\qquad (+ \cdots +) \\ &= \cdots + \underbrace{c}_{?} x^n y^n \end{aligned}$$

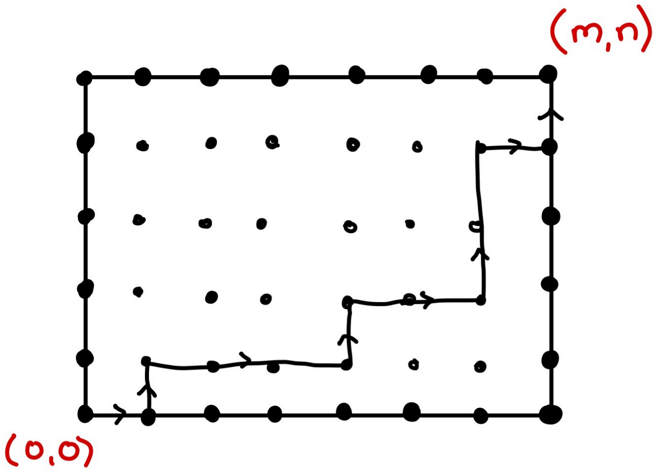
$$x^i y^{n-i} \cdot x^{n-j} y^j = x^{n+i-j} y^{n-i+j} = x^n y^n \text{ iff } i = j$$

Then the coefficient c in $cx^n y^n$ has to be

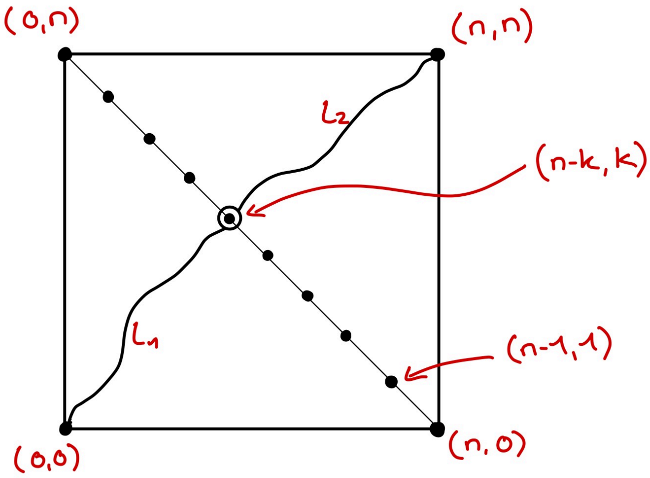
$$c = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{i} = \sum_{i=0}^{n} \binom{n}{i}^2$$

To summarize, we have three ways to prove combinatorial identities
- combinatorial
- by formula, put in the formula and calculate
- $\underbrace{\text{by algebraic identity}}_{\text{method of generating fns}}$

Example 1.29: rectangle, staircase path, many paths with same shortest length. Each step either up or to the right, otherwise not shortest path. Length is always $m - n$. Note whether up or right i.E. u, r, r, u. # Staircase paths = $\binom{m+n}{m}$

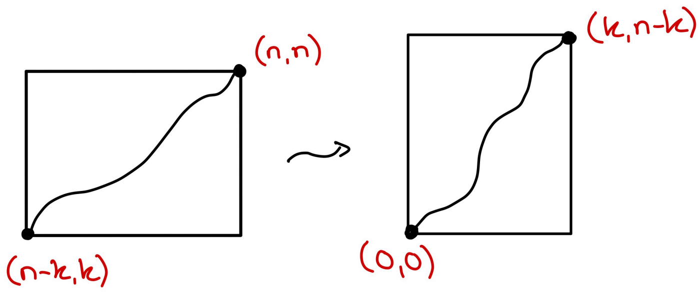


Now consider a square.



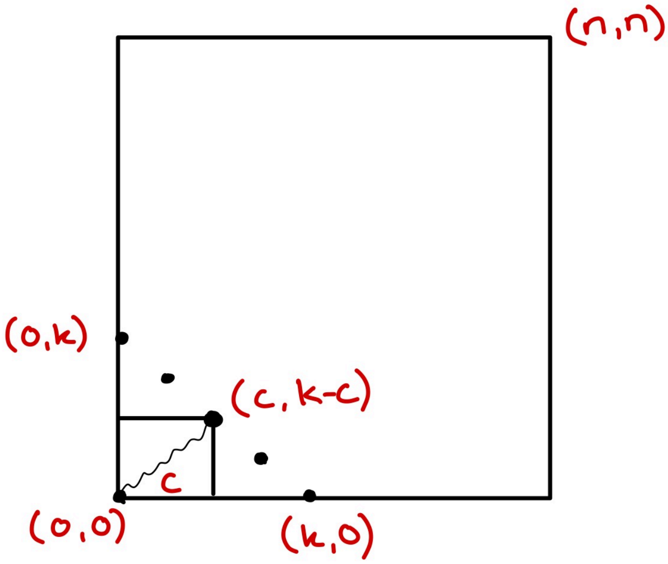
Now shortest path is of length $\binom{2n}{n}$. Have to pass exactly one checkpoint. Therefore each path contains a unique checkpoint $(n-k, k)$. $c_k = \#\text{paths through } (n-k, k)$ then $\binom{2n}{n} = \sum_{k=0}^{n} c_k = \sum_{k=0}^{n} \binom{n}{k}^2$. Each path contains the part of getting to the checkpoint with length l_1 and the part from the checkpoint to the destination of length l_2.

For l_1 the length is easy to calculate $l_1 : \binom{n}{k}$ways For l_2 we consider:

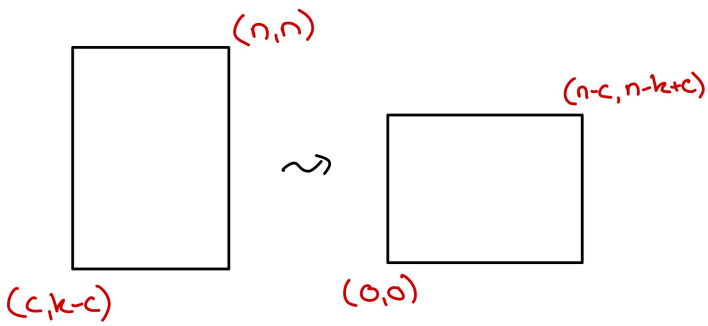


Therefore $l_2 : \binom{n}{k}$ways.

Further we consider a square with the checkpoints off the diagonal

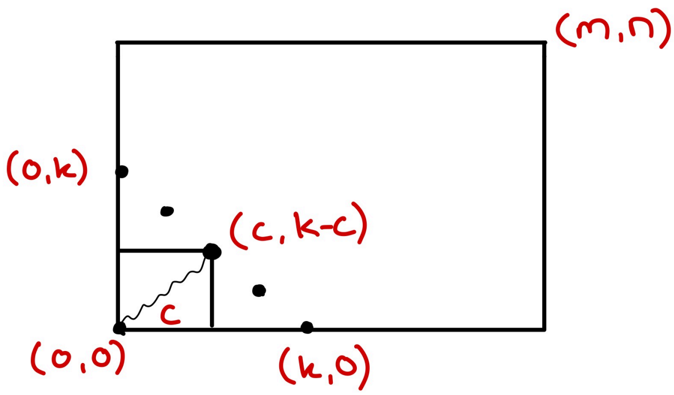


Here the same argument holds:

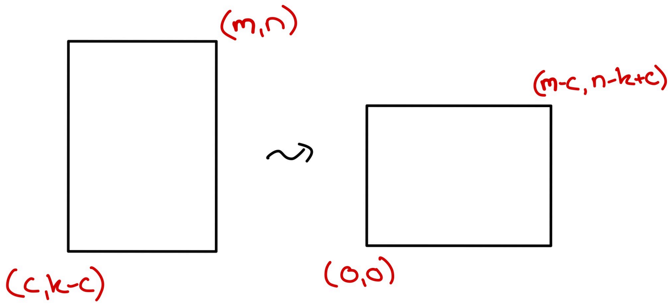


$$\binom{2n}{n} = \sum_{c=0}^{k} \binom{k}{c} \binom{2n-k}{n-c}$$

We now realize that this argument can be extended to general rectangles:



$$\binom{m+n}{n} = \sum_{c=0}^{k} \binom{k}{c} \binom{m+n-k}{m-c}$$



Special case: $n = k \leq m$

1.3 Multinomial coefficients

Definition 1.30: Let $n \geq 0, k_1, ..., k_t \geq 0, t \geq 1$ such that $n = k_1 + ... + k_t$. Then the *multinomial coefficient* is defined as

$$\binom{n}{k_1, ..., k_t} := \frac{n!}{k_1! \cdot ... \cdot k_t!}$$

Remark 1.31: for $t = 2$: $\binom{n}{k_1, k_2} = \frac{n!}{k_1! k_2!} = \binom{n}{k_1}$ for $n = k_1 + k_2 = k + n - k$ binomial coefficients: special case
$\binom{n}{k_1, ..., k_t}$ does not depend on the order of $k_1, ..., k_t$. for $t \geq 2$:

$$\binom{n}{k_1, ..., k_{t-1}, 0} = \binom{n}{k_1, ..., k_{t-1}}$$

If a k_i is 0, it can be discarded:

$$\binom{n}{k_1, ..., k_{t-1}, 0} = \binom{n}{k_1, ..., k_{t-1}}$$

If a k_i is 1, the following holds:

$$\binom{n}{k_1, ..., k_{t-1}, 1} = n\binom{n-1}{k_1, ..., k_{t-1}}$$

Further:

$$\binom{n}{k_1, ..., k_t} = \binom{n}{k_t}\binom{n-k_t}{k_1, ..., k_{t-1}}$$

Theorem 1.32:

$$\binom{n}{k_1, ..., k_t} = \#\text{ways to distribute } n \text{ objects into } t \text{ bins}$$

Proof: Let $S = \#$ ways to distribute n objects into bins $1, ..., t$ such that
(1) k_i objects land in bin i
(2) objects are ordered within the bin

$|S| = c \cdot k_1! \cdot ... \cdot k_t!$ There exists a bijection $S \longleftrightarrow$ all permutations of $[n]$ $|S| = n!$ ■

We now adapt Theorem 1.25 to multinomials

Theorem 1.33 (Multinomial theorem):

$$(x_1 + x_2 + \dots + x_t)^n = \sum_{k_1+\dots+k_t=n} c_{k_1,\dots,k_t} x_1^{k_1} \cdot x_2^{k_2} \cdots x_t^{k_t}$$

$$c_{k_1,\dots,k_t} = \binom{n}{k_1, \dots, k_t}$$

Proof: Same argument as in the proof of Theorem 1.25

$$(x_1 + \dots + x_t)^n = \underbrace{(x_1 + \dots + x_t)(\dots)\dots(x_1 + x_2 + \dots + x_t)}_{n}$$

■

Just like for binomial coefficients there are many formulas.

Lemma 1.34 (Formulas for multinomial coefficent):
(1) $k_m(k_1, \dots, k_m, \dots, k_n) = n\binom{n-1}{k_1,\dots,k_{m-1},\dots,k_t}$ for $k_m \geq 1$.
(2) $\binom{n}{k_1,\dots,k_t} = \binom{n-1}{k_1-1,k_2,\dots,k_t} + \binom{n-1}{k_1,k_2-1,k_3,\dots,k_t} + \binom{n-1}{k_1,k_2,\dots,k_t-1}$ *(Multi-analog of Pascal's triangle)*

Proof:

(1) Can also be proven in many different ways.
  - Directly using the definition
  - Combinatorial argument such as double counting: Choose a leader in set #m???????

(2)
  - By formula if you dare 😈
  - Combinatorial, just record in which subset is element n

■

Proposition 1.35:

$$\binom{n}{k_1, \dots, k_t} = \binom{n}{k_1} \cdot \binom{n-k_1}{k_2} \cdot \dots \cdot \binom{n - k_1 - \dots - k_{t-1}}{k_t}$$

Proof:

(1) Just use the formula™

$$\frac{n!}{k_1!(n-k_1)!} \cdot \frac{(n-k_1)!}{k_2!(n-k_1-k_2)!} \cdot \frac{(n-k_1-k_2)!}{k_3!(n-k_1-k_2-k_3)!} \cdot \dots$$
$$= \frac{n!}{k_1!\cancel{(n-k_1)!}} \cdot \frac{\cancel{(n-k_1)!}}{k_2!\cancel{(n-k_1-k_2)!}} \cdot \frac{\cancel{(n-k_1-k_2)!}}{k_3!\cancel{(n-k_1-k_2-k_3)}} \cdot \dots$$
$$= \frac{n!}{k_1!\dots k_t!}$$

(2) combinatorially choose the outset in a sequence

$$\binom{n}{k_1, \dots, k_t}$$

does not depend on permutation of $k_1, \dots, k_t$ Let us compute $\binom{n}{l, k \cdot l, n-l}$ by using the prior formula

$$\binom{n}{l, k-l, n-l} = \binom{n}{l} \cdot \binom{n-l}{k-l}$$

On the other hand compute

$$\binom{n}{k-l, l, n-k} = \binom{n}{k-l} \cdot \binom{n-k+l}{l}$$

$$\binom{n}{n-k, l, k-l} = \binom{n}{k} \cdot \binom{k}{l}$$

■

1.4 Lucas Theorem

DFK was thinking about fractions over the weekend. In particular he was thinking about $\frac{24}{15}$. This is equal to $\frac{8}{5}$, which is also equal to $\frac{2}{1} \cdot \frac{4}{5}$. The obvious question now becomes: for what fractions does this work?

Example 1.36:

$b = 2$ Binary presentation

$b = 10$ Decimal presentation

$b = 16$ Hexadecimal presentation

b prime p-adic presentation

General base b $n = n_0 + n_1 b + n_2 b^2 + \dots + 0 \le n_i \le b_1 \forall i$ The numbers $n_0, n_1, \dots$ are called *digits*

Theorem 1.37 (Lucas theorem): *Let p be a prime number, $n = \sum_{i=0}^{\infty} n_i p^i$, $k = \sum_{i=0}^{\infty} k_i p^i$. Then the following holds*

$$\binom{n}{k} = \binom{(\dots n_2 n_1 n_0)_p}{(\dots k_2 k_1 k_0)_p} \equiv \prod_{p \ge 0} \binom{n_i}{k_i} \bmod p$$

$$\equiv \binom{n_0}{k_0} \binom{n_1}{k_1} \dots \bmod p$$

Example 1.38:
- Compute $\binom{2025}{995}$ and $\binom{2025}{1000}$ mod 7.

$$2025 = 5 \cdot 343 + 6 \cdot 49 + 27 + 2 = (5622)_7$$
$$995 = (2621)_7$$

Therefore, by Lucas theorem we get

$$\binom{2025}{995} = \binom{(5622)_7}{(2621)_7} \equiv \binom{5}{2} \cdot \binom{6}{6} \cdot \binom{2}{2} \cdot \binom{2}{1} \bmod 7 \equiv 32 = \bmod 7$$

aswell as

$$\binom{2025}{1000} = \binom{(5622)_7}{(2626)_7} = \binom{5}{2} \cdot \binom{6}{6}\binom{2}{2} \cdot \binom{2}{6} \equiv 0 \bmod 7$$

Remark 1.39: What happens if n is a prime number? That is, we want to compute the binomial coefficient $\binom{p}{k}$ where p is prime and $k \leq p$. Then $p = (10)_p$ and $k = (ab)_p$, where either $a = 0$ or $a = 1$. Then by Lucas we get

$$\binom{p}{k} \equiv \binom{1}{a} \cdot \binom{0}{b} \bmod p = 1 \cdot \binom{0}{b}$$

For $b \neq 0$, the second term is equal to zero, and one for $b = 1$. Thus we get

$$\binom{p}{k} \equiv \begin{cases} 1 \bmod p & \text{for } b = 0 \Longleftrightarrow k \in \{0, p\} \\ 0 \bmod p & \text{else} \end{cases}$$

More generally, by the same argument we can compute $\binom{p^\alpha}{k} \bmod p$ for $\alpha \in \mathbb{N}$. We have

$$\binom{p^\alpha}{k} \bmod p \equiv 0 \bmod p \quad \text{unless } k = 0 \text{ or } k = p^\alpha$$

The previous remark implies the following result

Lemma 1.40 (Freshman's dream): *For a prime number p we have the following identity $(x + y)^p = x^p + y^p \bmod p$*

Lemma 1.41: *Let p be a prime, $n \geq k \geq 0$. Assume $n = a \cdot p + r$ and $k = bp + q$ where $0 \leq r, q \leq p - 1$. Then*

$$\binom{ap + r}{bp + q} = \binom{n}{k} \equiv \binom{a}{b} \cdot \binom{r}{q} \bmod p$$

Proof:

$$\begin{aligned}(1 + t)^n = (1 + t)^{ap+r} = (1 + t)^{ap}(1 + t)^r &= ((1 + t)^p)^{a(1+t)^r} \\ &\underset{\text{fresh}}{\overset{\bmod p}{\equiv}} (1 + t^p)^{a(1+t)^r} \\ &= \sum_{i=0}^{a} \binom{a}{i} t^{pi} \sum_{j=0}^{r} \binom{r}{j} t^j\end{aligned}$$

Considering the coefficient of $t^k = t^{bp+q}$, there is only one term in the product. Namely we must choose $i = b$ and $j = q$

$$\binom{a}{b} \cdot \binom{r}{q} \equiv \binom{n}{k} \bmod p$$

■

Proof Lucas theorem: Set $r = n_0$, $a := (\ldots n_2 n_1)_p$, $q := k_0$ and $b := (\ldots k_2, k_1)_p$

$$\binom{(\ldots n_2 n_1 n_0)_p}{(\ldots k_2 k_1 k_0)_p} \equiv \binom{(\ldots n_2 n_1)_p}{(\ldots k_2 k_1)_p} \binom{n_0}{k_0} \equiv \text{and so on } 🤡$$

■

Example 1.42 (Pascals triangle):

(1) mod 2 Pascal

$$\begin{array}{ccccccccccccccc}
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
& 1 & & 0 & & 1 & & 0 & & 1 & & 0 & & 1 & \\
& & 1 & & 1 & & 0 & & 0 & & 1 & & 1 & & \\
& & & 1 & & 0 & & 0 & & 0 & & 1 & & & \\
& & & & 1 & & 1 & & 1 & & 1 & & & & \\
& & & & & 1 & & 0 & & 1 & & & & & \\
& & & & & & 1 & & 1 & & & & & & \\
& & & & & & & 1 & & & & & & &
\end{array}$$

When is $\binom{n}{k}$ odd?
Let

$$n = (\ldots n_2 n_1 n_0)_2 \quad n_i \in \{0, 1\}$$
$$k = (\ldots k_2 k_1 k_0)_2 \quad k_i \in \{0, 1\}$$
$$\binom{n}{k} = \prod \binom{n_i}{k_i} \bmod 2$$

as $\binom{1}{1} = \binom{0}{0} = \binom{1}{0} = 1$, $\binom{0}{1} = 0$

Criterion for parity of $\binom{n}{k}$: $\binom{n}{k}$ is odd iff binary form of n "dominates" the binary form of k.

(2) mod 3 :

1 1 0 2 2 0 1 1

1 0 0 2 0 0 1

1 2 1 1 2 1

1 1 0 1 1

1 0 0 1

1 2 1

1 1

1

1.5 Binomial transform

Definition 1.43 (Sequence transformation): Let $\{\vartheta_n\}_{n\geq 0}$ be a sequence of numbers its binomial transform is $\{t_n\}_{n\geq 0}$ defined by

$$t_n = \sum_{k=0}^{n} \binom{n}{k} \varphi_k, \forall n \geq 0\,.$$

Definition 1.44 (Pascal matrix): This is basically the matrix you get if you arrange the pascal triangle in the lower triangle. For $d \in \mathbb{N}_0$, P_d is a $(d+1) \times (d+1)$-matrix. The rows and columns are indexed by the numbers $0, 1, \ldots, d$ and the entries are given by $P_d(\lambda_{r,c})_{0\leq r,c\leq d}$ with $\lambda_{r,c} := \binom{r}{c}$.

We now want to find the inverse of the pascal matrix. Therefore, we define another matrix.

Definition 1.45: We define the matrix $Q_d = (\beta_{r,c})_{0\leq r,c\leq d}$ as a $(d+1) \times (d+1)$-matrix with entries $\beta_{r,c} := (-1)^{r+c}\binom{r}{c}$.

Example 1.46:

(1) The Pascal matrix P_4 is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}$$

(2) The matrix Q_4 is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}$$

Theorem 1.47 (Pascal inversion): *For all* $d \geq 0 : P_d^{-1} = Q_d$ *.For all* $d \geq 0 : P_d^{-1} = Q_d$*.*

Corollary 1.48: *Let* $\{\varphi_n\}_{n\geq 0}$ *be a sequence of numbers, and* $\{t_n\}_{n\geq 0}$ *its binomial transform. then*

$$\mathfrak{s}_n \sum_{(-1)^{k+n}}^{n} \binom{n}{k} t_n \,.$$

Proof: $\vec{\mathfrak{s}}_d = \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_d \end{pmatrix}$ and $\vec{t}_d = \begin{pmatrix} t_0 \\ t_1 \\ \vdots \\ t_d \end{pmatrix}$

$$\begin{aligned} P_d \cdot \vec{\mathfrak{s}}_d &= \vec{t}_d \\ \vec{\mathfrak{s}}_d &= P_d^{-1} \vec{t}_d \\ &= Q_d \vec{t}_d \end{aligned}$$

■

Proof Theorem 1.47: $P_d^{-1} = Q_d$. then show $P_d Q_d = \mathbb{1}$

$P_d \cdot Q_d = (\gamma_{r,c})_{0\leq r,c \leq d}$

$$\gamma_{r,c} = \sum_{k=0}^{d} \alpha_{r,f} \cdot \beta_{k\cdot c} = \sum_{k=0}^{d} (-1)^{k+c} \binom{r}{k} \cdot \binom{k}{c}$$

$$\binom{r}{k}\binom{k}{c} = 0$$

unless $r \geq k \geq c$ so if $r < c \Rightarrow \gamma_{r,c} = 0$

$\boldsymbol{r = c}$

$\gamma_{r,r} = (-1)^{2r}\binom{r}{r}\binom{r}{r} = 1$ idk $r = k = c$

$\boldsymbol{r > c}$

$$\begin{aligned} \gamma_{r,c} = \sum_{k=0}^{d} (-1)^{k+c} \binom{r}{k}\binom{k}{c} &= \sum_{k=c}^{r} (-1)^{k+c} \binom{r}{k}\binom{k}{c} \\ &= \sum_{k=c}^{r} (-1)^{k+c} \binom{r}{c}\binom{r-c}{k-c} \\ &= \binom{r}{c} \sum_{k=c}^{r} (-1)^{k-c} \binom{r-c}{k-c} \\ &\overset{(\text{i=k}-\text{c})}{=} \binom{r}{c} \underbrace{\sum_{i=0}^{r-c} (-1)^{i} \binom{r-c}{i}}_{=0} = 0\,. \end{aligned}$$

■

Remark 1.49: $P_d = \exp(A_d) = e^{A_d}$ where A_d is defined as the matrix

$$\begin{pmatrix} 0 & 0 & 0 & \ldots & & 0 \\ 1 & 0 & 0 & \ldots & & 0 \\ 0 & 2 & 0 & \ldots & & 0 \\ 0 & 0 & 3 & \ldots & & 0 \\ & \ldots & & \ldots & & 0 \\ & \ldots & & 0 & d & 0 \end{pmatrix}$$

$$e^{A_d} = I + A_d + \frac{A_d^2}{2} + \ldots$$

Example 1.50: $A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, A_1^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ then

$$e^{A_1} = I_2 + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

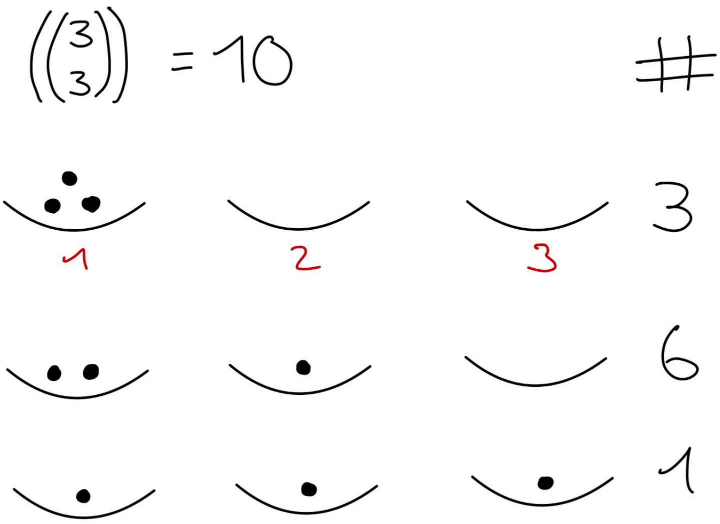
2 Combinatorial sequences

Selections with repetitions

Definition 2.1: Let $t, k \in \mathbb{N}_0$ and $t \geq 1$. We define $\left(\!\binom{t}{k}\!\right)$ as the number of selections of k *objects* out of t *types*.

Alternatively you can describe it as placing k identical objects into t bins, where each bin represents a type.

Example 2.2:
- $\left(\!\binom{1}{k}\!\right) = 1 = \binom{k}{0}, \forall k$
- $\left(\!\binom{2}{k}\!\right) = k + 1 = \binom{k+1}{1}$
-



- $\left(\!\binom{3}{k}\!\right) = \sum_{i=0}^{k}(k - i + 1) = \frac{(k+1)(k+2)}{2} = \binom{k+2}{2}$

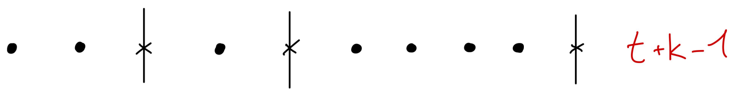
Theorem 2.3:

$$\left(\!\binom{t}{k}\!\right) = \binom{t+k-1}{t-1} = \binom{t+k-1}{k}$$

Proof: Can be proved by induction or directly combinatorially. Here we prove it by the method of dots and dividers. Instead of choosing k dots to be placed in t bins, we choose $t - 1$ dividers and assign the dots correspondingly. We claim that there is a bijection between choosing $t - 1$ dividers out of $t + k - 1$ dots and placing k identical objects into t bins.

■

Remark 2.4 (Method of dots & dividers):



choose $t - 1$ out of $t + k - 1$

Proposition 2.5: *The number $\left(\binom{t}{k}\right)$ counts the presentations of k as a sum of t nonnegative numbers.*

Proof: There exits a bijection between writing k as a sum of t nonegative numbers and placing k identical objects into t bins. ∎

Example 2.6:
$\boldsymbol{t = k = 3}$ $3 = 2 + 0 + 1 = 2 + 1 + 0 = 0 + 3 + 0 = \ldots$

$\sqrt{2\pi n} \cdot \frac{n}{e^n}$

2.1 Stirling numbers of the the 2nd kind

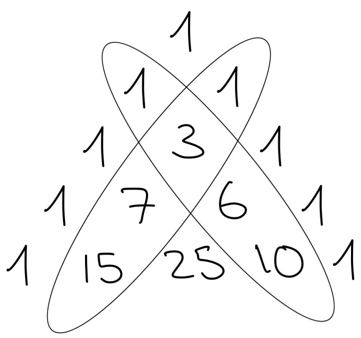
Definition 2.7: Let n and k be integers, $n \geq k \geq 1$.

$$S(n,k) := \#\text{ of ways to partition } [n] \text{ into } k \text{ non-empty parts.}$$

Example 2.8: $S(5,3) = \binom{5}{3} + 5 \cdot 3 = 10 + 15 = 25$

Convention 2.9: $S(n,0) = \begin{cases} 0 \text{ if } n \geq 1 \\ 1 \text{ if } n = 0 \end{cases}$

Stirling triangle



Example 2.10:
- $S(n,1) = S(n,n) = 1$
- $S(n,2) = 2^{n-1} - 1$ since $[n] = A \cup ([n] \setminus A)$ gives us $\frac{1}{2}(2^n - 2)$
- $S(n, n-1) = \binom{n}{2}$

Proposition 2.11: *For $n \geq k \geq 1$ we have the following recursion:*

$$S(n+1,k) = k \cdot S(n,k) + S(n,k-1)$$

(compare to binominal coefficient)

Proof: $S(n+1,k) = \#$ ways to partition $[n+1]$ into k nonempty parts. Further $[n+1] = \{1, 2, \ldots, n+1\} = [n] \cup \{n+1\}$. Let π be such a partition

$$\pi = \pi_1, ..., \underbrace{\pi_k}_{\ni n+1}$$

Case 1 $\pi_k = \{n+1\}$ $\# = S(n, k-1)$
Case 2 $|\pi_k| \geq 2$, then $\pi_1, \pi_2, ..., \pi_{k-1}, \pi_k \setminus \{n+1\}$ still non-empty. Bijection to all partitions of $[n]$ into k non-empty parts $S(n, k)$ with one part special: we put $n+1$ there.
total $S(n, k-1) + k \cdot S(n, k)$

■

Proposition 2.12 (Non-recursive formula for $S(n, k)$):

$$\begin{aligned} S(n,k) &= \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^n \\ &= \frac{1}{k!} \left(k^n - \binom{k}{k-1} (k-1)^n + ... + (-1)^{k-2} \binom{k}{2} 2^n + (-1)^{k-1} \binom{k}{1} 1^n \right) \end{aligned}$$

2.2 Falling factorials basis

Definition 2.13: For $k \geq 1$ we define

$$(x)_k := x(x-1)(x-2)...(x-k+1) = \frac{n!}{(n-k)!}$$

which are called <u>falling factorial</u> of x. $(n)_k = k!\binom{n}{k}$

- $(x)_1 = x$
- $(x)_2 = x(x-1)$
- $(x)_3 = x(x-1)(x-2)$
- ...

Vector space of polynomials of degree $\leq d$ dim $= d+1$

$$a_0 + a_1 x + ... + a_d x^d \quad \mathbb{R}^{d+1}$$

coordinates: $(a_0, a_1, ..., a_d)$ a monomial basis $1, x, x^2, ..., x^d$ Falling factorials also form a basis for this space

$$\begin{aligned} (x)_0 &:= 1 \\ (x)_1 &= x \\ (x)_2 &= x^2 - x \\ (x)_3 &= x^3 + 3x^2 + 2x \end{aligned}$$