6.5/20

Advanced Algorithms UE 4

by Maarten Behn
Group 5

Exercise 4.1

> a) Give an example of a network $\mathcal{N}$ in which $|V| \leq 4$ and the number of iterations of the Ford-Fulkerson Algorithm is c_{max}, where c_{max}, is the largest capacity of any arc.
> b) What is the size of the input for this example?
> c) Prove that this leads to an exponential running time (w.r.t. the size of the input).

a)

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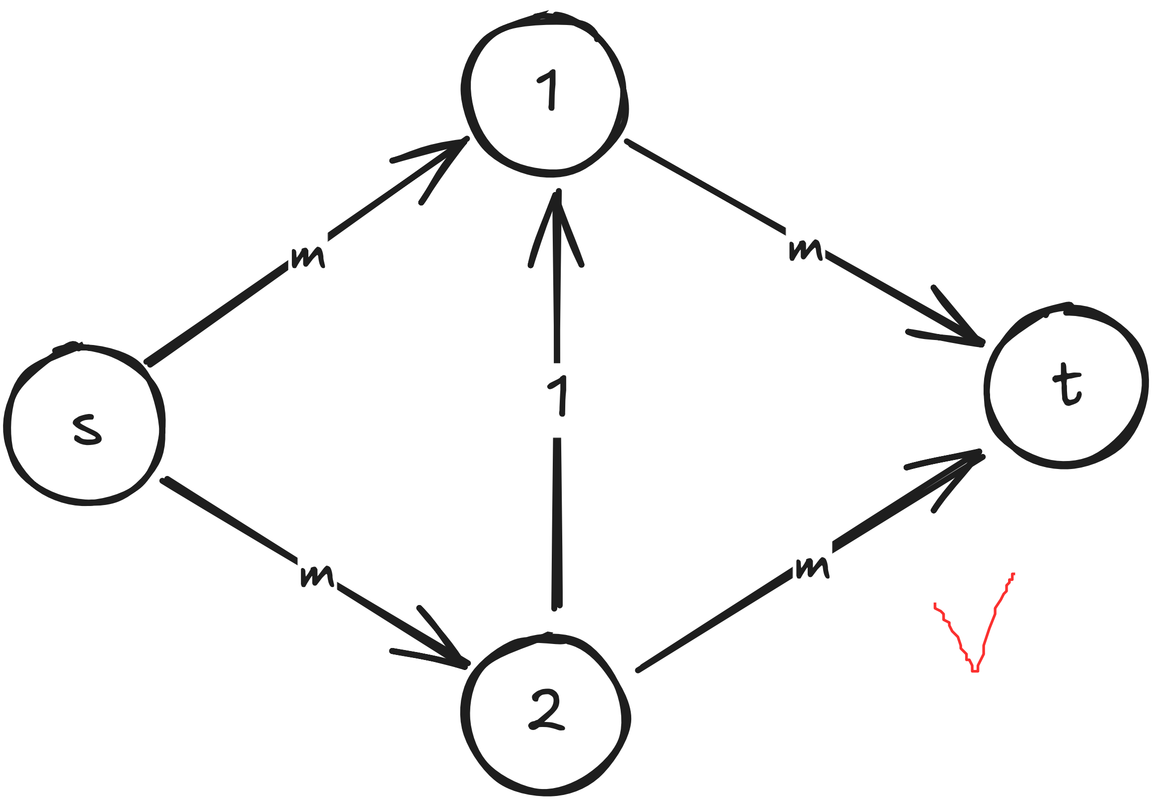


$m \in \mathbb{N}$ and therefore $m = c_{max}$

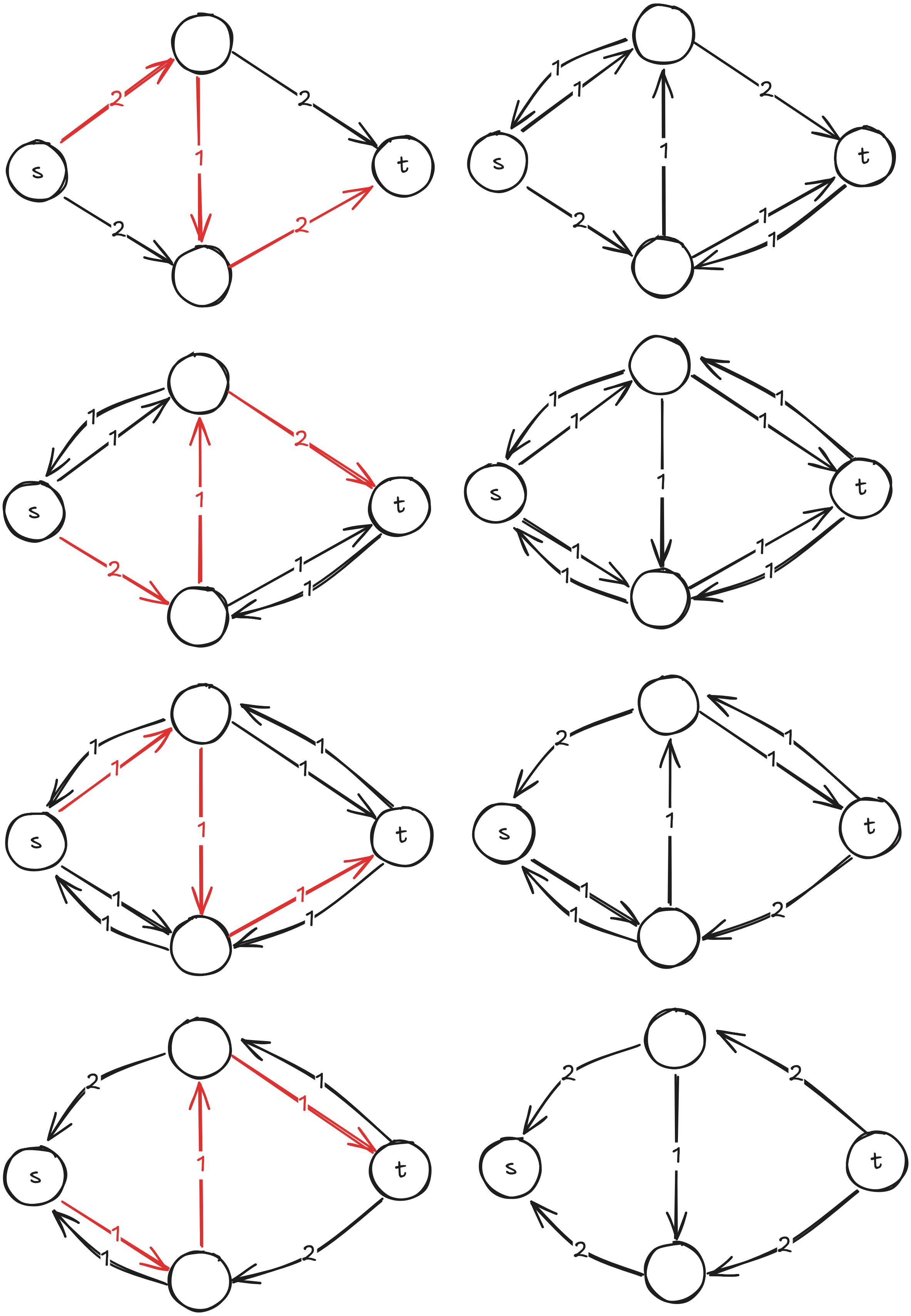
$\mathbb{N} := \mathbb{N} \backslash \{0\}$

Example for $m = 2$

"Worst" paths

If the Algorithm always happens to choose the path over the center edge
the algorithm takes $2 \cdot m$ Iterations.

How to change the instance to have precisely m iterations?



Ford-Fulkerson always choose an path over the center edge.

The incoming flow to t is increaes by one for every iteration.
t has an incoming flow of $2 \cdot m$. Therefore Ford-Fulkerson takes up to

$2 \cdot m$ iterations for this exmaple.

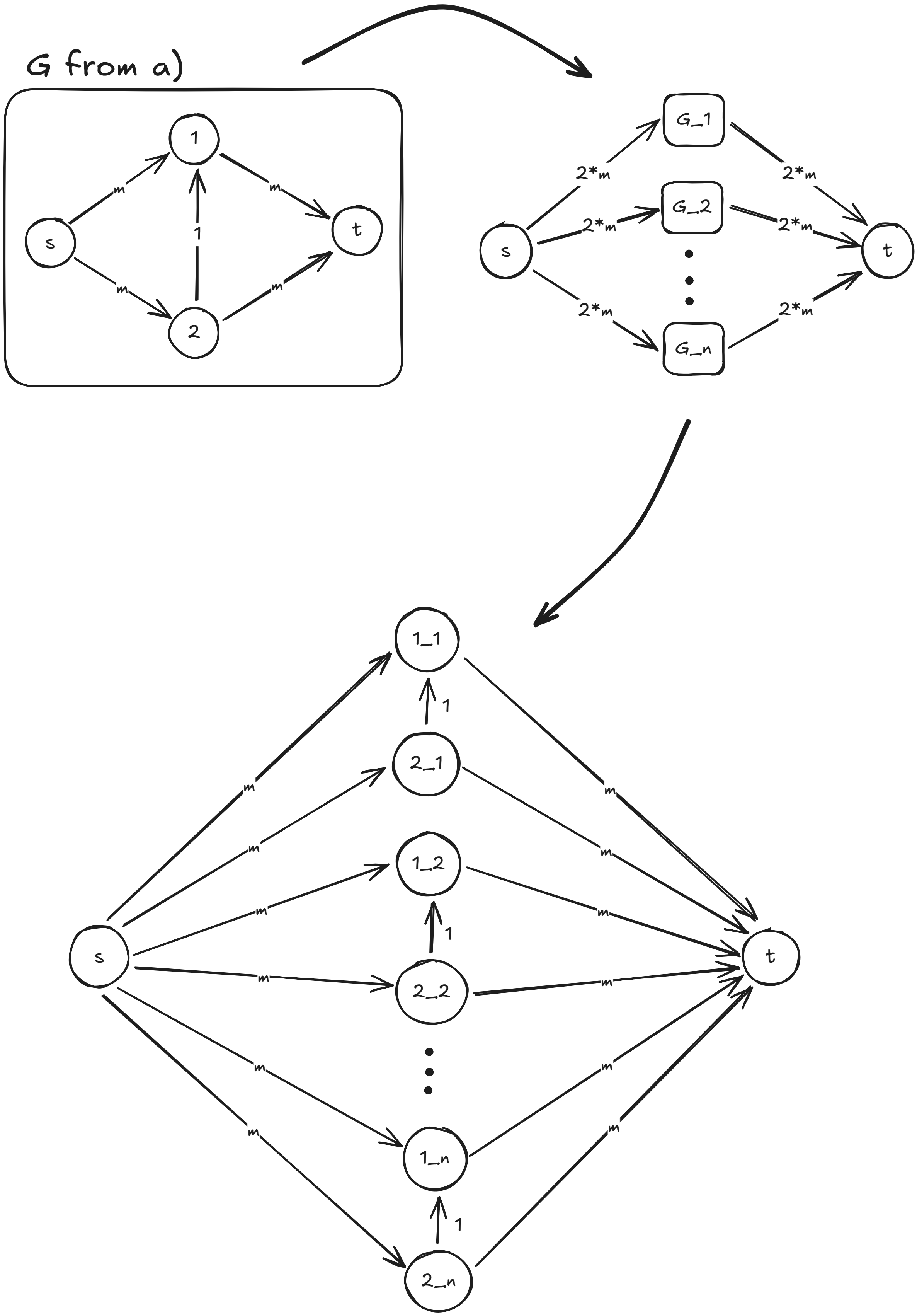
Proof that Ford-Fulkerson always choose an path over the center edge.

Ford-Fulkerson repeatly chooses s-2-1-t and s-1-2-t.
After every s-2-1-t path 1-2 has a capacity of 1 and s-1-2-t can be choosen next.
After every s-1-2-t path 2-1 has a capacity of 1 and s-2-1-t can be choosen next.

$2 \cdot m \leq c_{max}$

b)

Use the Graph from a) to construct:



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The input size is $m \cdot n$

We ask for the size of the example from a). Moreover what is n and m? you also have to take into account the value of c_max,which you can represent with bits.

c)

I proved in a) that Ford-Fulkerson can take up to $2 \cdot m$ Iterations for a sub graph G.
In the Graph of b) consist of n Sub graphs of a).

Therfore the G of b) has $|A| = 5 \cdot n$ edges.

1/2 In the Lecture the runtime of Ford-Fulkerson is $O(|A| \cdot M)$ with M value of maximum flow.
For G of b) $M = 2 \cdot n \cdot m$

Therfore G of b) has a Runtime of
$O(5 \cdot n \cdot 2 \cdot n \cdot m) = O\left(n^2 \cdot m\right)$

which corresponds to a Runtime of
$O\left(\left(\frac{|A|}{5}\right)^2 \cdot c_{max}\right) = O\left(|A|^2 \cdot c_{max}\right)$

Exercise 4.2

> Let f be a flow in a network $\mathcal{N} = (V, E, c, s, t)$, and let g be a flow in the residual network $\mathcal{N}_f$ of $\mathcal{N}$ w.r.t. f.
> Prove the following statements:
> a) $f + g$ is a flow in $\mathcal{N}$.
> b) If g is a maximum flow in $\mathcal{N}_f$, then $f + g$ is a maximum flow in $\mathcal{N}$.

a)

Define the sum of flows as $t = f + g$
$t(u, v) = f(u, v) + g(u, v) - g(v, u)$ what is (u,v)?
and
$c_t(u, v) = c(u, v) - f(u, v)$

capacity constraints

For $u, v \in V \quad u \neq v$ what if (u,v) is not an edge?

show that
$0 \leq t(u,v) \leq c(u,v)$

since $f(u,v) \leq c(u,v)$ and $g(u,v) \leq c_t(u,v)$?
$f(u,v) + g(u,v) \leq c(u,v)$

since $g(v,u) \leq 0$
$f(u,v) + g(u,v) - g(v,u) \leq f(u,v) + g(u,v)$

$t(u,v) \leq c(u,v)$

since $f(u,v) \geq 0$ and $g(v,u) \leq f(u,v)$ (there is only an backwards residual egde if there is a flow $f(u,v) > 0$)
$t(u,v) \geq f(u,v) - g(v,u) \geq 0$

flow conservation

For $v \in V \backslash \{s,t\}$

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since f and g hold the flow conservation
$\sum_u f(u,v) = \sum_u f(v,u)$
$\sum_u g(u,v) = \sum_u g(v,u)$

?

$\sum_u t(u,v) = \sum_u f(u,v) - \sum_u g(u,v) - \sum_u g(v,u) = \sum_u f(u,v) - \sum_u g(u,v) - \sum_u g($

what is u?

what is happening here?

b)

Since $f+g$ is a flow -> a)

Proof by contradiction

Suppose that t is not a maximum flow in N. Then, there exists some flow h in N such that $|h| > |f + g|$ what does |h| mean?

1/4 $|h| - |f| > |f + g| - |f| = |g|$?

this is a contradiction with g being maximum.

Since g is a maximum flow in the residual network N_f, the Optimality criteria Theorem says:
The residual network N_f has no augmenting s-t-paths.

Hence, by the Max-Flow Min-Cut Theorem, t is a maximum flow in N.

Exercise 4.3

> Let $N = (V, E, c, s, t)$ be an s-t-network with m edges.
> Prove that there exists a maximum flow in N that is the sum of at most m flows $f_1, \ldots, f_k$, each of which takes positive values only on a single s-t-path in N .
> Moreover, prove that if all capacities in N are integers, then $f_1, \ldots, f_k$ can be chosen as integral flows.

Proof by induction

Assumption

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$$f = \sum_{i=0}^{k} f_i + \sum_{i=0}^{n} c_i$$

where

1. f_i is either an s-t-path and c_i an flow cicle. This means only the edges on the path or cicle have positive value.
2. $k \leq |E|$
3. f_i is an integral flow when f is an integral flow.

Step repeat while $|f'| \neq 0$ what is f'?

Define a residual graph G_f based on f'
G_f contains only edges with a positive flow $f'(e) > 0$

Pick an vertex v with an non zero incoming flow.
Search for an s-t-path P covering v.
If there is no path then there must be a circle C that covers v.

$\delta = min_{e \in \text{P or C}} f'(e)$

Construct a new flow f_P or f_C from P or C where the value of all edges is δ
$f' := f' - f_P \text{ or } f_C$

Proof $k \leq |E|$ unclear

There can be at most $|E|$ s-t-paths in G
And there is no path f_a, f_b that are equal, because the edge with the minimum capacity is has a capacity of zero after the path of f_a is found.

Proof f_i is an integral flow when f is an integral flow.

If f is intigral δ is always intigral therefore f_i is intigral.

Source:
https://theory.stanford.edu/~trevisan/cs261/lecture11.pdf (11.05.'25 17:20)