Advanced Algorithms UE 7

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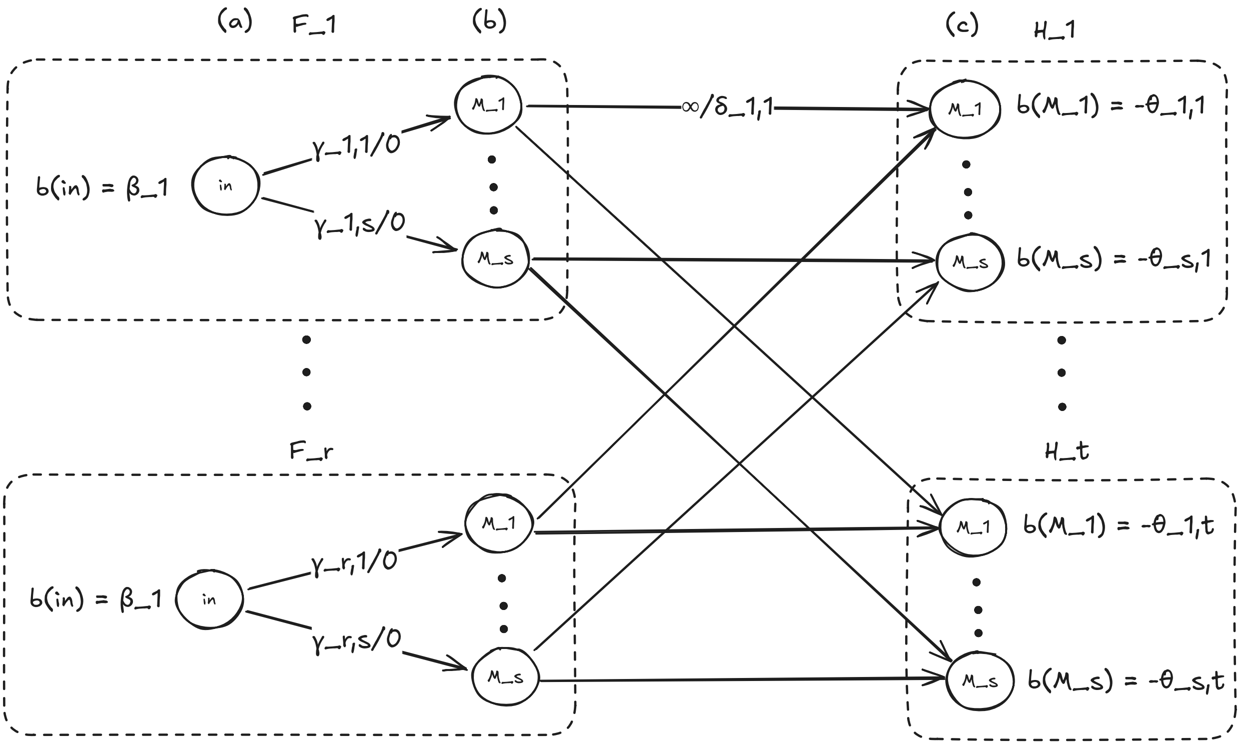
Exercise 7.1 (Modeling as Min-Cost Flow) (10 Points)

> Consider the following problem.
> A car manufacturer has r factories $F_1, \ldots, F_r$ to produce s different car models $M_1, \ldots, M_s$.
> For $i \in [r]$ and $j \in [s]$, it is known whether factory F_i can produce model Mj and, if so, at what unit cost $\alpha_{i,j}$.
>
> Furthermore, factory F_i for $i \in [r]$ can produce at most β_i cars per month, of which at most $\gamma_{i,j}$ can be of model M_j , $j \in [s]$.
>
> The cars are sold by t car dealers $H_1, \ldots, H_t$.
> For $i \in [r]$ and $k \in [t]$, it costs $\delta_{i,k}$ to transport a vehicle from F_i to H_k
> .
>
> Additionally, dealer H_k, for $k \in [t]$, sells $\theta_{j,k}$ cars of model $M_j \in [s]$ per month.
> The goal is to plan the production and transportation of cars such that the total cost is minimized.
>
> Model the above optimization problem as a minimum-cost flow problem.
> Specify the set of nodes and edges, as well as the edge capacities, costs, and balances.
> Argue why your modeling solves the given production problem.

Idea



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Formal definition

$$\mathcal{N} = (V, E, c, b, p)$$

$$V = \{a_i | i \in [r]\}$$
$$\cup \{b_{i,j} | i \in [r] j \in [s]\}$$
$$\cup \{c_{j,k} | j \in [s] k \in [t]\}$$

✓

~~only add the edge if factory~~ i can produce ~~model j~~

$$E = \{(a_i, b_{i,j}) | i \in [r] j \in [s]\}$$ ~~-1~~
$$\cup \{(b_{i,j}, c_{j,k}) | i \in [r] j \in [s] k \in [t]\}$$

$$b(a_i) = \beta_i$$
$$b(b_{i,j}) = 0$$
$$b(c_{j,k}) = -\theta_{j,k}$$

The sum of all the b-values should equal 0, which is not the case for your b-function. -2

$$c((a_i, b_{i,j})) = \gamma_{i,j}$$
$$c(b_{i,j}, c_{j,k})) = \infty$$

✓

$$p((a_i, b_{i,j})) = 0$$ alpha_{ij} -1
$$p(b_{i,j}, c_{j,k})) = \delta_{i,k}$$

Argument why a min const flow problem on $\mathcal{N}$ models the optimization problem

Each Factory has in a_i an input flow corresponding to β_i.
This models the amount of cars each factory can produce.

This flow gets split up into $b_{i,1} \dots b_{i,s}$.
The edges have a max capacity of $\gamma_{i,j}$
If the factory does not produce the model the capacity would be 0. ✓
Therefore incoming flow into $b_{i,j}$ represents how many cars of model j factory i can produce.

The flow from $b_{i,j}$ gets split up into $c_{j,1} \dots c_{j,t}$
The incoming flow into $c_{j,k}$ represents how many cars of model j dealer k sells.
Therefore the $b(c_{j,k})$ is $-\theta_{j,k}$

The edges $p((b_{i,j}, c_{j,k}))$ have a cost of $\delta_{i,k}$ to represent the cost of transporting a car of model j from factory i to dealer k.
Therefore the edges $(b_{i,j}, c_{j,k})$ represent the distribution problem of cars to dealers.

Exercise 7.2 (Decomposition into Cycles) (10 Points)

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> Prove the following lemma from the lecture:
> Let f be a circulation in a network $N = (V, E, c, b, p)$. Then, there exist flows $f_1, \dots, f_k$ with $k \leq m$ such that
> (i) $f = f_1 + \dots + f_k$,
> (ii) f_i is a feasible circulation in $\mathcal{N}$ for all $i \in [k]$, and
> (iii) f_i takes positive values only on edges of a cycle C_i in N for all $i \in [k]$.

Idea

repeatedly finding a cycle in f and extracting flow along these cycle until
the f is empty.

Helper definition

$E_f := \{e \in E \mid f(e) > 0\}$
$G_f := (V, E_f)$

Observation

When f is a non empty circulation there must be at least one cycle in G_f.

Proof

Start at $(u, v_1) \in E_f$
since $\sum_{e \in \delta^-(v_1)} = \sum_{e \in \delta^+(v_1)} \exists (v_1, v_2) \in E_f$
what is this? -1
following all possible edges recursively must lead back to v_1 this proof is circular -2
forming a cycle
because G_f is finite and all flow in a circulation is conserved.

Algorithm

While f is not empty
Create G_f for f
Let C be a cycle found in G_f
$\forall e \in C \quad f_i(e) := min(f(e))$ where are you taking the minimum over? this notation is insufficient -1
$\forall e \notin C \quad f_i(e) := 0$
$f \leftarrow f - f_i$ You should also mention that f-f_i is still a circulation, hence you can iteratively find cycles, not just in the first iteration. -1

Proof f_i is a feasible circulation

what do you mean? -1

- satisfies conserves flow -> same amount goes in a circle.
- satisfies capacity constraints -> all flow is positive and less or equal to the original f

Proof $k \leq m$

Each iteration removes at least one edge form E_f (the one with the minimum flow on the cycle)
so $k \leq |E| = m$

You should learn to write complete sentences, starting on a line, and ending with a point. Having a lot of line breaks and incomplete sentences makes it hard to understand. You are loosing points in the formulation, while I think you understand it very well.