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Advanced Algorithms UE 9

by Maarten Behn
Group 5

Exercise 9.1 (MST Modeling and Separation) (8 Points)

> Consider the Minimum Spanning Tree (MST) problem over a graph $G = (V, E)$ with edge costs c_e and variables $x_e \in \{0, 1\}$ indicating whether edge e is in the solution.
>
> Use the so-called subtour elimination constraint
>
> $$\sum_{e \in E(S)} x_e \leq |S| - 1 \ \ \forall S \subset V, 2 \leq |S| \leq |V| - 1$$
>
> and one global constraint:
>
> $$\sum_{e \in E} x_e = |V| - 1$$
>
> (a) Give an ILP formulation for the MST problem and argue why it is correct.

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Variables:

$x_e \in \{0, 1\} \ \ \forall e \in E$

Optimization function

$$min \sum_{e \in E} c_e x_e$$

Constraints

$$\sum_{e \in E(S)} x_e \leq |S| - 1 \quad \forall S \subset V, 2 \leq |S| \leq |V| - 1$$

$$\sum_{e \in E} x_e = |V| - 1$$

$$x_e \in \{0, 1\} \quad \forall e \in E$$

Argument why the IPL correct is

Feasibility enforces spanning trees:

- The cardinality constraint guarantees that the selected edges form a subgraph with exactly $|V| - 1$ edges.
- The subtour elimination constraints prevent any cycle from forming in any subset S of vertices. Since any cycle would violate the inequality

$$\sum_{e \in E(S)} x_e \leq |S| - 1,$$

  no cycles can exist in the chosen edge set.

Connectivity is ensured implicitly:

- Given that the graph has $|V|$ vertices and $|V| - 1$ edges, and no cycles are allowed (due to subtour constraints), the selected edges form a forest.
- Since the cardinality is exactly $|V| - 1$ edges, the forest cannot be disconnected (a disconnected forest would have fewer than $|V| - 1$ edges).
- Therefore, the selected edges form a connected acyclic subgraph — i.e., a spanning tree.

3. Optimality of the solution:

- By minimizing the sum of costs

$$\sum_{e \in E} c_e x_e,$$

the ILP finds the spanning tree with minimum total edge cost.

> (b) Give a polynomial-time separation algorithm for the LP-relaxation of your ILP.
> Hint: Use your knowledge about flow or cut problems.

We want to separate the subtour elimination constraints from a given fractional solution $\mathbf{x} = (x_e)_{e \in E}$ of the LP-relaxation, where $x_e \in [0, 1]$. That is, we want to check if there exists any subset $S \subset V$ with $2 \leq |S| \leq |V| - 1$ such that

$$\sum_{e \in E(S)} x_e > |S| - 1,$$

which would violate the subtour elimination constraint.

Note that

$$\sum_{e \in E(S)} x_e \leq |S| - 1$$

is equivalent to

$$\sum_{e \in \delta(S)} x_e \geq 1,$$

where $\delta(S)$ are the cut edges of S.

The total number of edges inside S plus the edges crossing the cut $\delta(S)$ relate to the degrees and connectivity. Since $\sum_{e \in E} x_e = |V| - 1$, the subtour elimination can equivalently be detected by cuts of low x_e-weight.

Therefore the subtour elimination constraints can be replaced by cut constraints:

$$\sum_{e \in \delta(S)} x_e \geq 1, \quad \forall S \subset V, S \neq \emptyset, V.$$

Goal of the Algorithm

Given fractional values x_e, find a subset S such that

$$\sum_{e \in \delta(S)} x_e < 1.$$

If such a set S exists, then the constraint corresponding to S is violated.

Algorithm

Input: fractional solution $x_e \in [0, 1]$.

- For each vertex $v \in V$:
  - Construct graph with edge capacities $c_e = x_e$.
  - For each other vertex $w \neq v$, compute the minimum v-w cut.
  - If min cut value < 1, output the cut S inducing the violation.
- If no cut with capacity < 1 found, no subtour elimination constraint is violated.

Proof

- The minimum cut problem can be solved in polynomial time via max-flow algorithms.
- Since the number of vertices is $|V|$, and for each v we consider cuts to all other vertices, the total number of max-flow computations is polynomial (at most $|V| \times (|V| - 1)$).
- Each violated subtour elimination corresponds to a cut with capacity less than 1.
- This separation algorithm finds such cuts efficiently or confirms no violations exist.

Source of the Idea

https://theory.epfl.ch/osven/courses/Approx13/Notes/lecture9and10.pdf (22.06.2025, 21:37)

Exercise 9.2 (Total Unimodularity) (4 Points)

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> Let $G = (V, E)$ be a bipartite graph.
> Prove that the node-edge incidence matrix of G is totally unimodular.
> Hint: Use one of the properties stated in class.

Setup

Let $M \in \{0,1\}^{|V| \times |E|}$ node-edge incidence matrix of G.
$m_{v,e} = 1$ e is indecent to v and 0 is it is not. ?
M is in row (the vertices), column (the edges) order.

Let $V_1, V_2 \subseteq V$ be the two sides of the bipartite Graph.
(all edges have one vertex in V_1 and one in V_2)

Observation

An edge is always connected to two vertices.

$$\Rightarrow \forall e \in E \sum_{v \in V} m_{v,e} = 2$$

=> Every column in M contains two ones.

Characteristics of determiant

from Mathe 1 Course

Given a quadratic Matrix M
construct M' from M by multiplying k rows by -1 then:

$$det(M') = (-1)^k \cdot det(M) = \begin{cases} det(M) & \text{if } \ k \mid 2 \\ -det(M) & \text{if } \ k \nmid 2 \end{cases}$$

Proof:
https://proofwiki.org/wiki/Determinant_with_Row_Multiplied_by_Constant?utm_source=chatgpt.com

Proof

Let A be quadratic sub matrix of M.
Construct A' from A by multiplying all rows belonging to vertices from S_2 by -1.
Now every column in A' contains at most one $+1$ and one -1.
The Lemma by Poincar´e states that A' is unimodular.
Then A is also unimodular because multiplying rows -1 does not change the fact that the $det(A) \in \{-1, 0, 1\}$.

Exercise 9.3 (Complementary Slackness) (8 Points)

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> Dualize the following LPs and test with complementary slackness whether the given solutions are optimal.

Primal LP (P):

$$
\begin{array}{rll}
\max & x_1 + 3x_2 + x_3 & \\
\text{s.t.} & x_1 + 2x_2 + 7x_3 \le -3 & \text{1. Constraint} \\
& x_2 - x_3 = 9 & \text{2. Constraint} \\
& 9x_1 \le 5 & \text{3. Constraint} \\
& x_1 \ge 0 & \text{4. Constraint} \\
& x_2 \le 0 & \text{5. Constraint}
\end{array}
$$

Dual variables:

- y_1 for constraint 1 ($\le$): $y_1 \ge 0$
- y_2 for constraint 2 ($=$): unrestricted
- y_3 for constraint 3 ($\le$): $y_3 \ge 0$

Dual LP (D):

$$
\begin{array}{rl}
\min & -3y_1 + 9y_2 + 5y_3 \\
\text{s.t.} & y_1 + 9y_3 \ge 1 \\
& 2y_1 + y_2 \ge 3 \quad \le \qquad -1 \\
& 7y_1 - y_2 \ge 1 \quad = \\
& y_1 \ge 0, y_3 \ge 0, \; y_2 \in \mathbb{R}
\end{array}
$$

Given Primal Solution:

$$x = (0, 0, -9)^T$$

1. $x_1 + 2x_2 + 7x_3 = 0 + 0 + 7(-9) = -63 \le -3$
2. $x_2 - x_3 = 0 - (-9) = 9$
3. $9x_1 = 0 \le 5$
4. $x_1 = 0 \ge 0$
5. $x_2 = 0 \le 0$

   All constraints are satisfied

Given Dual Solution:

$$y = \left(\frac{4}{9}, \frac{19}{9}, \frac{1}{9}\right)^T$$

1. $y_1 + 9y_3 = \frac{4}{9} + 9 \cdot \frac{1}{9} = \frac{13}{9} > 1$
2. $2y_1 + y_2 = 2 \cdot \frac{4}{9} + \frac{19}{9} = \frac{27}{9} = 3$
3. $7y_1 - y_2 = 7 \cdot \frac{4}{9} - \frac{19}{9} = \frac{28-19}{9} = 1$
4. $y_1 = \frac{4}{9} \geq 0$
5. $y_3 = \frac{1}{9} \geq 0$

   All constraints are satisfied

Complementary Slackness Fails:

- Constraint 1 is slack $(-63 < -3)$ → $y_1 = 0$ → No
- Constraint 3 is slack $(9x_1 = 0 < 5)$ → $y_3 = 0$ → No

Primal and dual feasible, but complementary slackness fails, so the solutions are not optimal.

b)

Primal LP (P):

$$\begin{aligned} &\min 5x_1 + 6x_2 \\ &\text{s.t. } 3x_1 - x_2 \geq 8 && \text{1. Constraint} \\ &\quad 2x_1 + 4x_2 = -2 && \text{2. Constraint} \\ &\quad 3x_1 + 2x_2 \leq 4 && \text{3. Constraint} \\ &\quad x_{1 \geq 0} && \text{4. Constraint} \end{aligned}$$

Dual variables:

- y_1 for constraint 1 ($\geq$): $y_1 \geq 0$
- y_2 for constraint 2 ($=$): unrestricted
- y_3 for constraint 3 ($\leq$): $y_3 \leq 0$

Dual LP:

$$\begin{aligned} \max\ & 8y_1 - 2y_2 + 4y_3 \\ \text{s.t. } & 3y_1 + 2y_2 + 3y_3 \le 5 \\ & -y_1 + 4y_2 + 2y_3 \le 6 \\ & y_1 \ge 0,\ y_3 \le 0,\ y_2 \in \mathbb{R} \end{aligned}$$

-1

$=$

Given Primal Solution:

$$x = \left(\frac{15}{7}, -\frac{11}{7}\right)^T$$

1. $3x_1 - x_2 = 3 \cdot \frac{15}{7} - (-\frac{11}{7}) = \frac{45+11}{7} = \frac{56}{7} = 8$
2. $2x_1 + 4x_2 = 2 \cdot \frac{15}{7} + 4 \cdot (-\frac{11}{7}) = \frac{30-44}{7} = -2$
3. $3x_1 + 2x_2 = 3 \cdot \frac{15}{7} + 2 \cdot (-\frac{11}{7}) = \frac{45-22}{7} = \frac{23}{7} \approx 3.29 \le 4$
4. $x_1 = \frac{15}{7} \ge 0$

All constraints are satisfied

Given Dual Solution:

$$y = \left(\frac{4}{7}, \frac{23}{14}, 0\right)^T$$

1. $3y_1 + 2y_2 + 3y_3 = 3 \cdot \frac{4}{7} + 2 \cdot \frac{23}{14} = \frac{12}{7} + \frac{46}{14} = \frac{12+23}{7} = \frac{35}{7} = 5$
2. $-y_1 + 4y_2 + 2y_3 = -\frac{4}{7} + 4 \cdot \frac{23}{14} = -\frac{4}{7} + \frac{92}{14} = -\frac{4}{7} + \frac{46}{7} = \frac{42}{7} = 6$
3. $y_1 = \frac{4}{7} \ge 0$
4. $y_3 = 0 \le 0$

All constraints are satisfied

Complementary Slackness passes:

- Constraint 1 is tight $\rightarrow$ y_1 can be $\neq 0$

- Constraint 2 is tight → y_2 unrestricted
- Constraint 3 is slack → $y_3 = 0$
- Variable $x_1 = \frac{15}{7} \neq 0$, so dual constraint 1 must be tight
- Variable $x_2 = -\frac{11}{7} \neq 0$, so dual constraint 2 must be tight

Primal and dual feasible, complementary slackness holds, so the solutions are optimal.